MATH 226: Notes on Assignment 4

Practice Problems 2.1: 1, 3, 7, 12, 13*, 16*, 17*, 21, 29, 33

1. Rewriting as $ydy = x^4dx$, then integrating both sides, we have $y^2/2 = x^5/5 + c$, or $5y^2 - 2x^5 = c$; $y \neq 0$

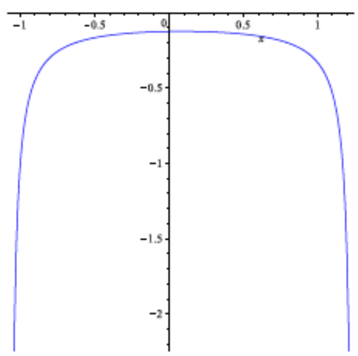
3. Rewriting as $y^{-3}dy = -\sin x dx$, then integrating both sides, we have $-y^{-2}/2 = \cos x + c$, or $y^{-2} + 2\cos x = c$ if $y \neq 0$. Also, $y = 0$ is a solution.

7. Rewriting as $(y/(1+y^2))dy = xe^{x^2}dx$, then integrating both sides, we obtain $\ln(1+y^2) = e^{x^2} + c$. Therefore, $y^2 = ce^{e^{x^2}} - 1$.

12. Rewriting as $dy/(y - y^2) = xdx$, then integrating both sides, we have $\ln|y| - \ln|1-y| = x^2/2 + c$, or $y/(1-y) = ce^{x^2/2}$, which gives $y = e^{x^2/2}/(c + e^{x^2/2})$. Also, $y = 0$ and $y = 1$ are solutions.

13.(a) Rewriting as $y^{-2}dy = (1 - 12x)dx$, then integrating both sides, we have $-y^{-1} = x - 6x^2 + c$. The initial condition $y(0) = -1/8$ implies $c = 8$. Therefore, $y = 1/(6x^2 - x - 8)$.

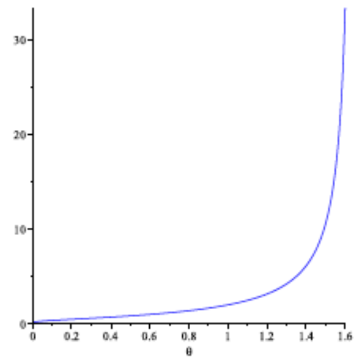
(b)



(c) $(1 - \sqrt{193})/12 < x < (1 + \sqrt{193})/12$

16.(a) Rewriting as $r^{-2}dr = \theta^{-1}d\theta$, then integrating both sides, we have $-r^{-1} = \ln|\theta| + c$. The initial condition $r(1) = 2$ implies $c = -1/2$. Therefore, $r = 2/(1 - 2\ln|\theta|)$.

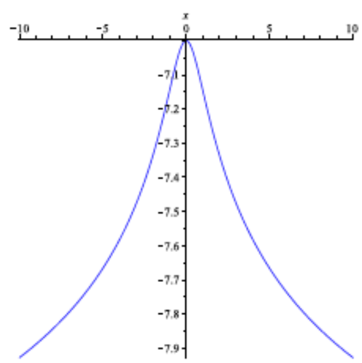
(b)



(c) $0 < \theta < \sqrt{e}$

17.(a) Rewriting as $ydy = 3x/(1 + x^2)dx$, then integrating both sides, we have $y^2/2 = 3\ln(1 + x^2)/2 + c$. The initial condition $y(0) = -7$ implies $c = 49/2$. Therefore, $y = -\sqrt{3\ln(1 + x^2) + 49}$.

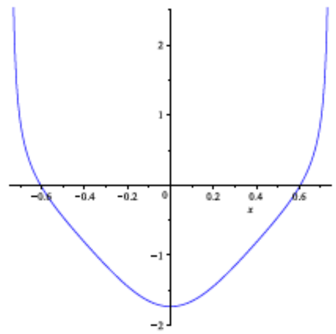
(b)



(c). The solution is valid for all real numbers *x*.

21.(a) Rewriting as $dy/(1+y^2) = \tan 2x dx$, then integrating both sides, we have $\arctan y = -\ln(\cos 2x)/2 + c$. The initial condition $y(0) = -\sqrt{3}$ implies $c = -\pi/3$. Therefore, $y = -\tan(\ln(\cos 2x)/2 + \pi/3)$.

(b)



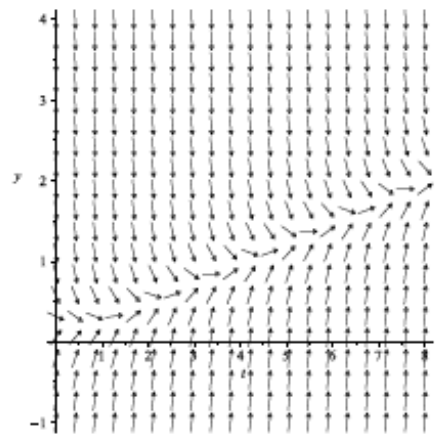
(c) $-\pi/4 < x < \pi/4$

29. Rewriting the equation as $(12y^2 - 12y)dy = (1 + 3x^2)dx$ and integrating both sides, we have $4y^3 - 6y^2 = x + x^3 + c$. The initial condition $y(0) = 2$ implies $c = 8$. Therefore, $4y^3 - 6y^2 - x - x^3 - 8 = 0$. When $12y^2 - 12y = 0$, the integral curve will have a vertical tangent. This happens when $y = 0$ or $y = 1$. From our solution, we see that $y = 1$ implies $x = -2$; this is the first y value we reach on our solution, therefore, the solution is defined for $-2 < x < \infty$.

33. Rewriting the equation as $(10 + 2y)dy = 2\cos 2x dx$ and integrating both sides, we have $10y + y^2 = \sin 2x + c$. By the initial condition $y(0) = -1$, we have $c = -9$. Completing the square, it follows that $y = -5 + \sqrt{\sin 2x + 16}$. To find where the solution attains its maximum value, we need to check where $y' = 0$. We see that $y' = 0$ when $2\cos 2x = 0$. This occurs when $2x = \pi/2 + 2k\pi$, or $x = \pi/4 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Practice Problems 2.2: 1*, 4*, 7*, 8*, 13, 15, 22*, 25*, 31, 35, 36, 38, 41

1.(a)



(b) All solutions seem to converge to an increasing function as $t \to \infty$.

(c) The integrating factor is $\mu(t) = e^{4t}$. Then

$$e^{4t}y' + 4e^{4t}y = e^{4t}(t + e^{-2t})$$

implies that

$$(e^{4t}y)' = te^{4t} + e^{2t},$$

thus

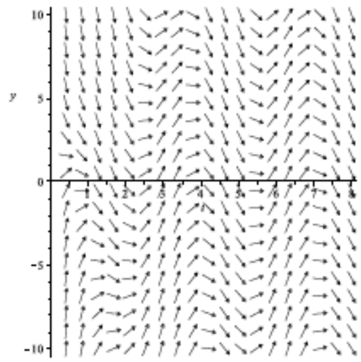
$$e^{4t}y = \int (te^{4t} + e^{2t})\,dt = \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t} + \frac{1}{2}e^{2t} + c,$$

and then

$$y = ce^{-4t} + \frac{1}{2}e^{-2t} + \frac{t}{4} - \frac{1}{16}.$$

We conclude that y is asymptotic to the linear function $g(t) = t/4 - 1/16$ as $t \to \infty$.

4.(a)



(b) The solutions eventually become oscillatory.

(c) The integrating factor is $\mu(t) = t$. Therefore, $ty' + y = 5t\cos 2t$ implies $(ty)' = 5t\cos 2t$, thus

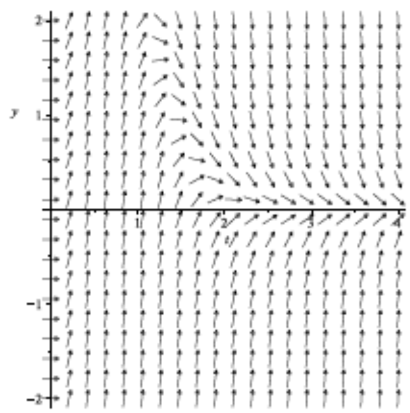
$$ty = \int 5t\cos 2t\,dt = \frac{5}{4}\cos 2t + \frac{5}{2}t\sin 2t + c,$$

and then

$$y = \frac{5\cos 2t}{4t} + \frac{5\sin 2t}{2} + \frac{c}{t}.$$

We conclude that y is asymptotic to $g(t) = (5\sin 2t)/2$ as $t \to \infty$.

7.(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $g(t) = 0$.

(c) The integrating factor is $\mu(t) = e^{t^2}$. Therefore,

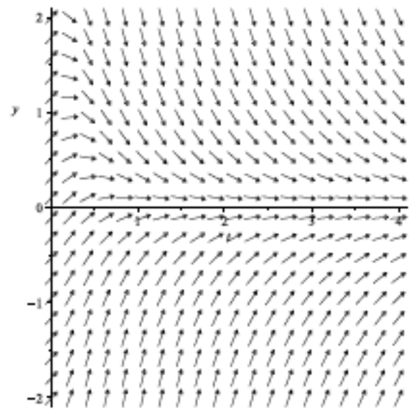
$$(e^{t^2} y)' = e^{t^2} y' + 2tye^{t^2} = 16t,$$

thus

$$e^{t^2} y = \int 16t \, dt = 8t^2 + c,$$

and then $y(t) = 8t^2 e^{-t^2} + ce^{-t^2}$. We conclude that $y \to 0$ as $t \to \infty$.

8.(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $g(t) = 0$.

(c) The integrating factor is $\mu(t) = (1+t^2)^2$. Then

$$(1+t^2)^2 y' + 4t(1+t^2)y = \frac{1}{1+t^2},$$

so

$$((1+t^2)^2 y) = \int \frac{1}{1+t^2}\, dt,$$

and then $y = (\arctan t + c)/(1+t^2)^2$. We conclude that $y \to 0$ as $t \to \infty$.

13. The integrating factor is $\mu(t) = e^{-t}$. Therefore, $(e^{-t}y)' = 2te^t$, thus

$$y = e^t \int 2te^t\, dt = 2te^{2t} - 2e^{2t} + ce^t.$$

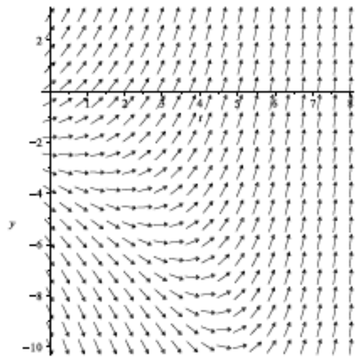
The initial condition $y(0) = 1$ implies $-2 + c = 1$. Therefore, $c = 3$ and $y = 3e^t + 2(t-1)e^{2t}$.

15. Dividing the equation by t, we see that the integrating factor is $\mu(t) = t^4$. Therefore, $(t^4 y)' = t^5 - t^4 + t^3$, thus

$$y = t^{-4} \int (t^5 - t^4 + t^3)\, dt = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + \frac{c}{t^4}.$$

The initial condition $y(1) = 1/4$ implies $c = 1/30$, and $y = (10t^6 - 12t^5 + 15t^4 + 2)/60t^4$.

22.(a)

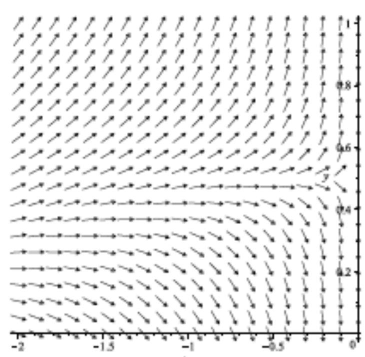


All solutions eventually increase or decrease without bound. The value a_0 appears to be approximately $a_0 = -3$.

(b) The integrating factor is $\mu(t) = e^{-t/2}$. From this, we get the equation $y'e^{-t/2} - ye^{-t/2}/2 = (ye^{-t/2})' = e^{-t/6}/2$. After integration, the general solution is $y(t) = -3e^{t/3} + ce^{t/2}$. The initial condition $y(0) = a$ implies $y = -3e^{t/3} + (a+3)e^{t/2}$. The solution will behave like $(a+3)e^{t/2}$. Therefore, $a_0 = -3$.

(c) $y \to -\infty$ for $a = a_0$.

25.(a)



It appears that $a_0 \approx .4$. That is, as $t \to 0$, for $y(-\pi/2) > a_0$, solutions will increase without bound, while solutions will decrease without bound for $y(-\pi/2) < a_0$.

(b) After dividing by t, we see that the integrating factor is $\mu(t) = t^2$. After multiplication by μ, we obtain the equation $t^2y' + 2ty = (t^2y)' = \sin t$, so after integration, we get that the general solution is $y = -\cos t/t^2 + c/t^2$. Using the initial condition, we get the solution $y = -\cos t/t^2 + \pi^2 a/4t^2$. Since $\lim_{t\to 0} \cos t = 1$, solutions will increase without bound if $a > 4/\pi^2$ and decrease without bound if $a < 4/\pi^2$. Therefore, $a_0 = 4/\pi^2$.

(c) For $a_0 = 4/\pi^2$, $y = (1 - \cos t)/t^2 \to 1/2$ as $t \to 0$.

31. The integrating factor is $\mu(t) = e^{-3t/2}$ and the general solution of the equation is $y(t) = -2t-4/3-4e^t+ce^{3t/2}$. The initial condition implies $y(t) = -2t-4/3-4e^t+(y_0+16/3)e^{3t/2}$. The solution will behave like $(y_0+16/3)e^{3t/2}$ (for $y_0 \neq -16/3$). For $y_0 > -16/3$, the solutions will increase without bound, while for $y_0 < -16/3$, the solutions will decrease without bound. If $y_0 = -16/3$, the solution will decrease without bound as the solution will be $-2t-4/3-4e^t$.

35. We notice that $y(t) = ce^{-t} + 4 - t$ approaches $4 - t$ as $t \to \infty$. We just need to find a first order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 4 - t$. Then $f' + f = 0$ and $g'+g = -1+4-t = 3-t$. Therefore, $y(t) = ce^{-t}+4-t$ satisfies the equation $y'+y = 3-t$. That is, the equation $y' + y = 3 - t$ has the desired properties.

36. We notice that $y(t) = ce^{-t} + 2t - 5$ approaches $2t - 5$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 2t - 5$. Then $f' + f = 0$ and $g' + g = 2 + 2t - 5 = 2t - 3$. Therefore, $y(t) = ce^{-t} + 2t - 5$ satisfies the equation $y' + y = 2t - 3$. That is, the equation $y' + y = 2t - 3$ has the desired properties.

38. Multiplying the equation by $e^{a(t-t_0)}$, we have $e^{a(t-t_0)}y + ae^{a(t-t_0)}y = e^{a(t-t_0)}g(t)$, so $(e^{a(t-t_0)}y)' = e^{a(t-t_0)}g(t)$ and then

$$y(t) = \int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds + e^{-a(t-t_0)}y_0.$$

Assuming $g(t) \to g_0$ as $t \to \infty$, and using L'Hôpital's rule,

$$\lim_{t\to\infty} \int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds = \lim_{t\to\infty} \frac{\int_{t_0}^{t} e^{as}g(s)\,ds}{e^{at}} = \lim_{t\to\infty} \frac{e^{at}g(t)}{ae^{at}} = \frac{g_0}{a}.$$

For an example, let $g(t) = 2 + e^{-t}$. Assume $a \neq 1$. Let us look for a solution of the form $y = ce^{-at}+Ae^{-t}+B$. Substituting a function of this form into the differential equation leads to the equation $(-A + aA)e^{-t} + aB = 2 + e^{-t}$, thus $-A + aA = 1$ and $aB = 2$. Therefore, $A = 1/(a-1)$, $B = 2/a$ and $y = ce^{-at} + e^{-t}/(a-1) + 2/a$. The initial condition $y(0) = y_0$ implies $y(t) = (y_0 - 1/(a-1) - 2/a)e^{-at} + e^{-t}/(a-1) + 2/a \to 2/a$ as $t \to \infty$.

41. Here, $p(t) = 1/t$ and $g(t) = 3\cos 2t$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) = e^{-\int \frac{1}{t}\,dt}\left(\int_0^t 3\cos 2\tau\, e^{\int \frac{1}{\tau}\,d\tau}\,d\tau + C\right) \\ &= \frac{1}{t}\left(\int_0^t 3\tau\cos 2\tau\,d\tau + C\right) = \frac{1}{t}\left(\frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t + C\right). \end{aligned}$$