MATH 226: Notes on Assignment 2

Practice Problems 1.1: 1, 3, 5, 7, 11, 13

1.(a) Let $u(t)$ be the temperature of the coffee at time t (measured in minutes), and $T_0 = 70$ be the ambient temperature. The initial value problem is $u' = -k(u - T_0)$, $u(0) = 200$.

(b) According to the text, the solution of the differential equation is $u(t) = T_0 + ce^{-kt}$. $200 = u(0) = T_0 + c$, thus $c = 130$. Also, $190 = u(1) = 70 + 130e^{-k}$, thus $e^{-k} = 12/13$. The coffee reaches temperature 170 when $170 = 70 + 130e^{-kt} = 70 + 130(12/13)^t$, thus $t = \ln(10/13)/\ln(12/13) \approx 3.28$ minutes.

3. Let $t = 0$ be 11:09pm, and let us measure time in hours. The temperature of the body is then given by $u(t) = 68 + 12e^{-kt}$. Also, $78.5 = u(1) = 68 + 12e^{-k}$. This gives that $e^{-k} = 7/8$. The time of death is given by the equation $98.6 = 68 + 12e^{-kt}$; we obtain that $t = \ln(51/20)/\ln(7/8) \approx -7.01$ hours, i.e. at around 4:08pm.

5.(a) The general solution is $p(t) = 900 + ce^{t/2}$. Plugging in for the initial condition, we have $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the solution is $p(t) = 900 - 50e^{t/2}$. To find the time when the population becomes extinct, we need to find the time T when $p(T) = 0$. Therefore, $900 = 50e^{T/2}$, which implies $e^{T/2} = 18$, and, therefore, $T = 2\ln 18 \approx 5.78$ months.

(b) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population will become extinct at the time T when $900 = (900 - p_0)e^{T/2}$. That is, $T = 2\ln[900/(900 - p_0)]$ months.

(c) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population after 1 year (12 months) will be $p(6) = 900 + (p_0 - 900)e^6$. If we want to know the initial population which will lead to extinction after 1 year, we set $p(6) = 0$ and solve for p_0. Doing so, we have $(900 - p_0)e^6 = 900$ which implies $p_0 = 900(1 - e^{-6}) \approx 897.8$.

7.(a) The general solution of the equation is $Q(t) = ce^{-rt}$. Given that $Q(0) = 100$, we have $c = 100$. Assuming that $Q(1) = 82.04$, we have $82.04 = 100e^{-r}$. Solving this equation for r, we have $r = -\ln(82.04/100) = 0.19796$ per week or $r = 0.02828$ per day.

(b) Using the form of the general solution and r found above, we have $Q(t) = 100e^{-0.02828t}$.

(c) Let T be the time it takes the isotope to decay to half of its original amount. From part (b), we conclude that $0.5 = e^{-0.2828T}$ which implies that $T = -\ln(0.5)/0.2828 \approx 24.5$ days.

11. Using the model from the text, we obtain the equation $Q' = (1 + (\sin t)/2)/2 - 2Q/100$, $Q(0) = 50$.

13. (a) $C(t) = C_0 e^{-kt}$; this function satisfies $dC/dt = -kC$ and $C(0) = C_0$.

(b) Observe that the concentration immediately after a dose is equal to (concentration immediately before the dose) + (the concentration C_0 of the dose).

At the end of the interval $[0,T]$, the concentration is $C_0 e^{-kT}$ so the concentration immediately after the dose at time T is $C_0 + C_0 e^{-kT} = C_0[1 + e^{-kT}]$.

At the end of the time period $[T,2T]$, the concentration would be $[C_0 + C_0 e^{-kT}] e^{-kT}$ since we have an initial value of $C_0 + C_0 e^{-kT}$ and it decays for a period of time T. Thus the concentration immediately after the dose at time $2T$ would be $Co + [C_0 + C_0 e^{-kT}] e^{-kT}$ which we write as $C_0 [1 + e^{-kT} + (e^{-kT})^2]$

Similarly, the concentration immediately after the dose at time $3T$ would be

$$C_0 + C_0 [1 + e^{-kT} + (e^{-kT})^2] e^{-kT} = C_0 [1 + e^{-kT} + (e^{-kT})^2 + (e^{-kT})^3]$$

(c) An induction argument shows that the concentration after the dose at time nT would be

$$C_0 [1 + e^{-kT} + (e^{-kT})^2 + (e^{-kT})^3 + \ldots + (e^{-kT})^n]$$

The term in square brackets is the start of a geometric series with first term 1 and common ratio e^{-kT}

As $n \to \infty$, The concentration approaches C_0 times the sum of the geometric series which is

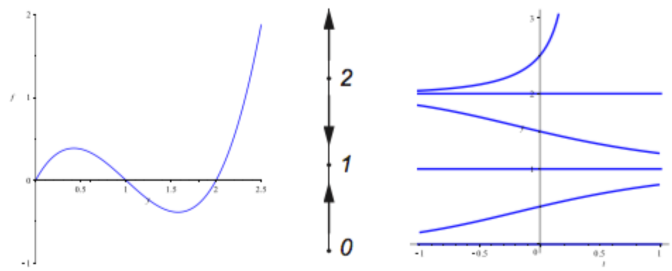
$$C_0 \frac{1}{1-e^{-kT}} = \frac{C_0}{1-e^{-kT}}$$

Recall that the sum of a geometric series with ($|r|< 1$) is

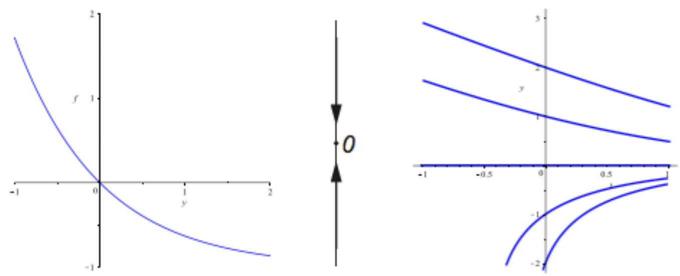
$$a + a^r + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$$

Practice Problems 1.2: 1, 3, 5, 13, 14, 15, 24-29

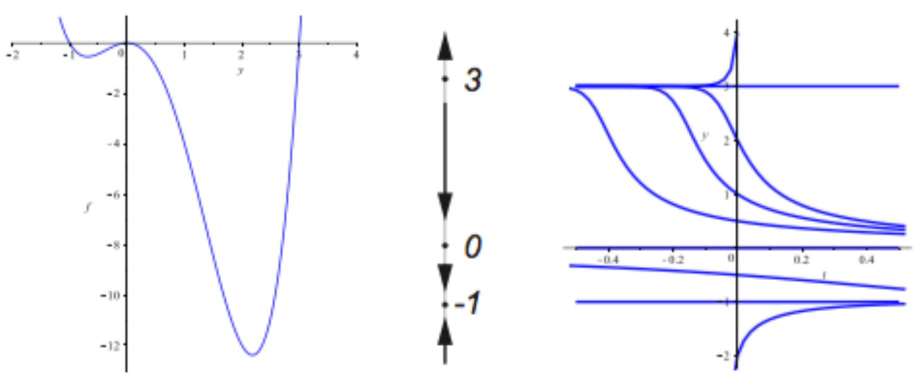
1. $y = 0$, $y = 2$ unstable, $y = 1$ asymptotically stable.



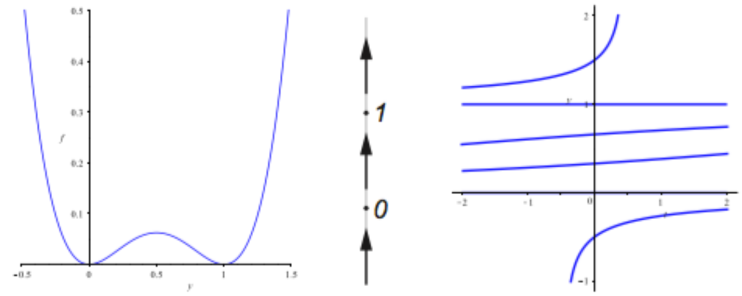
3. $y = 0$ asymptotically stable.



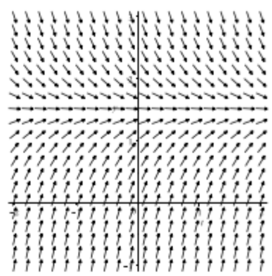
5. $y = -1$ asymptotically stable, $y = 0$ semistable, $y = 3$ unstable.



13. $y = 0$ semistable, $y = 1$ semistable.

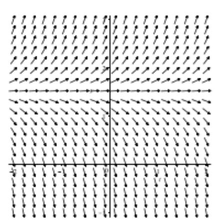


14.



For $y > 3/2$, the slopes are negative, thus the solutions decrease. For $y < 3/2$, the slopes are positive, thus the solutions increase. As a result, $y \to 3/2$ as $t \to \infty$ for all initial conditions y_0.

15.



For $y > 3/2$, the slopes are positive, therefore the solutions increase. For $y < 3/2$, the slopes are negative, therefore the solutions decrease. As a result, y diverges from $3/2$ as $t \to \infty$ if $y(0) \neq 3/2$.

24. (j) - only equilibrium is $y = 2$; $y' > 0$ when $y < 2$.

25. (c) - only equilibrium is $y = 2$; $y' < 0$ when $y < 2$.

26. (g) - only equilibrium is $y = -2$; $y' > 0$ when $y < -2$.

27. (b) - only equilibrium is $y = -2$; $y' < 0$ when $y < -2$.

28. (h) - equilibria at $y = 0$, $y = 3$; $y' > 0$ when $0 < y < 3$.

29. (e) - equilibria at $y = 0$, $y = 3$; $y' < 0$ when $0 < y < 3$.

1. The Fitzhugh-Nagumo model for the electrical impulse in a neuron states that, in the absence of relaxation effects, the electrical potential of a neuron $v(t)$ obeys the differential equation

$$\frac{dv}{dt} = -v(v^2 - (1+a)v + a)$$

where a is a positive constant such that $a \in (0,\ 1)$.

(a) Classify this differential equation. What are the independent and dependent variables? Any parameters?

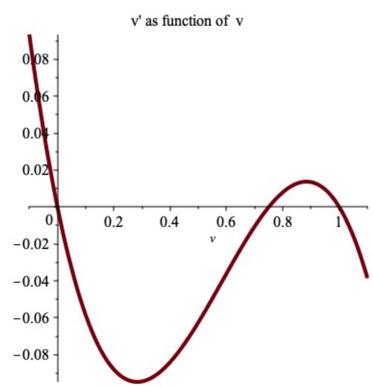
This is a first order, nonlinear differential equation. The independent value is t and the dependent value is v with parameter a

(b) For what values of v is v unchanging? Make sure to simplify as much as possible.

Factor the right hand side as - *v (v - a) (v - 1)* so v is unchanging at *v = 0, a, 1*

(c) For what values of v is v increasing? Decreasing? It may be helpful to draw a picture here.

dv/dt is positive for $v < 0$ and $a < v < 1$ so v increases in these regions; negative for $0 < v < a$ and $v > 1$ so v decreases in these regions



Assume that a neuron starts out with an electrical potential of -10 mV. According to this differential equation model, what will happen to the electrical potential as time goes to infinity? What if the neuron started out at $\frac{a}{2}$ mV instead?

In the first case, the potential will increase to 0; in the second, it will decrease to 0 .

Continuing your ideas from (d), explain all possible long-term outcomes for this neuron if it starts at different initial electrical potentials.

if initial potentials is below *a*, then it will converge to 0 and if it is above *a*, then it will converge to 1 .