MATH 226
Notes on Assignment 6

2.4 Differences Between Linear and Nonlinear Equations
Practice Problems 2.4: 1, 2, 4, 5, 7, 8, 14, 15, 16, 19*, 21*, 25, 27

1. Rewriting the equation as

$$y' + \frac{\ln t}{t-3} y = \frac{2t}{t-3}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $0 < t < 3$.

2. Rewriting the equation as

$$y' + \frac{1}{t(t-4)} y = 0$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $0 < t < 4$.

4. Rewriting the equation as

$$y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $-\infty < t < -2$.

5. Rewriting the equation as

$$y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $-2 < t < 2$.

7. Using the fact that

$$f = \frac{t-y}{2t+5y} \quad \text{and} \quad f_y = -\frac{7t}{(2t+5y)^2},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied as long as $2t + 5y \neq 0$.

8. Using the fact that

$$f = (1-t^2-y^2)^{1/2} \quad \text{and} \quad f_y = -\frac{y}{(1-t^2-y^2)^{1/2}},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied as long as $t^2 + y^2 < 1$.

14.(a) First, it is clear that $y_1(2) = -1 = y_2(2)$. Further,

$$y_1' = -1 = \frac{-t + (t^2 + 4(1-t))^{1/2}}{2} = \frac{-t + [(t-2)^2]^{1/2}}{2}$$

and

$$y_2' = \frac{-t + (t^2 - t^2)^{1/2}}{2}.$$

The function y_1 is a solution for $t \geq 2$. The function y_2 is a solution for all t.

(b) Theorem 2.4.2 requires that f and $\partial f/\partial y$ be continuous in a rectangle about the point $(t_0, y_0) = (2, -1)$. Since f_y is not continuous if $t < 2$ and $y < -1$, the hypotheses of Theorem 2.4.2 are not satisfied.

(c) If $y = ct + c^2$, then

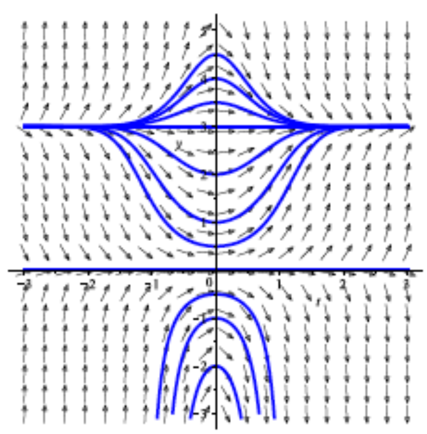
$$y' = c = \frac{-t + [(t+2c)^2]^{1/2}}{2} = \frac{-t + (t^2 + 4ct + 4c^2)^{1/2}}{2}.$$

Therefore, y satisfies the equation for $t \geq -2c$.

15. The equation is separable, $ydy = -4tdt$. Integrating both sides, we conclude that $y^2/2 = -2t^2 + y_0^2/2$ for $y_0 \neq 0$. The solution is defined for $y_0^2 - 4t^2 \geq 0$.

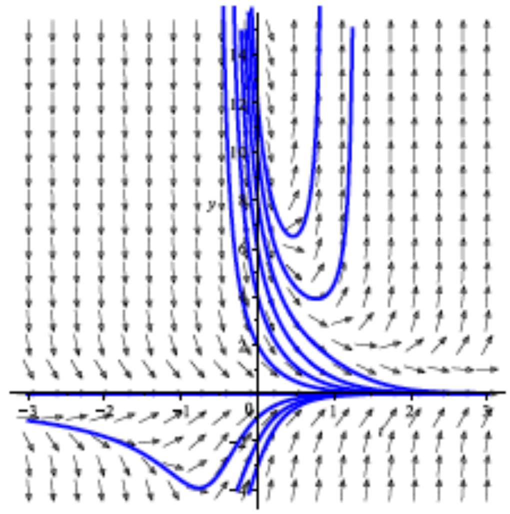
16. The equation is separable and can be written as $dy/y^2 = 2tdt$. Integrating both sides, we arrive at the solution $y = y_0/(1 - y_0 t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \leq 0$, solutions exist for all t.

19.



If $y_0 > 0$, then $y \to 3$. If $y_0 = 0$, then $y = 0$. If $y_0 < 0$, then $y \to -\infty$.

21.



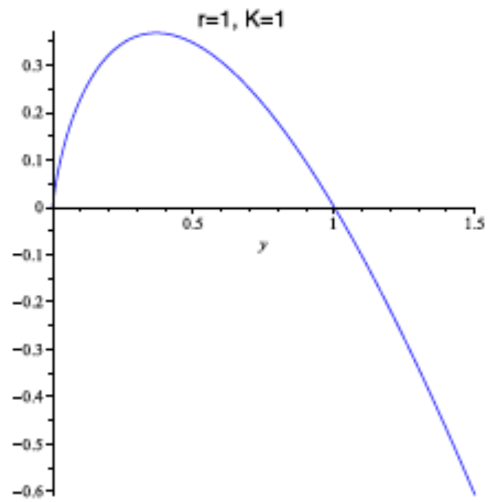
If $y_0 > 9$, then $y \to \infty$. If $y_0 \le 9$, then $y \to 0$.

25. Let $y = y_1 + y_2$, then $y' + p(t)y = y_1' + y_2' + p(t)(y_1 + y_2) = y_1' + p(t)y_1 + y_2' + p(t)y_2 = 0$.

27. The solution of the initial value problem $y' + 2y = 1$ is $y = 1/2 + ce^{-2t}$. For $y(0) = 0$, we see that $c = -1/2$. Therefore, $y(t) = \frac{1}{2}(1 - e^{-2t})$ for $0 \le t \le 1$. Then $y(1) = \frac{1}{2}(1 - e^{-2})$. Next, the solution of $y' + 2y = 0$ is given by $y = ce^{-2t}$. The initial condition $y(1) = \frac{1}{2}(1 - e^{-2})$ implies $ce^{-2} = \frac{1}{2}(1 - e^{-2})$. Therefore, $c = \frac{1}{2}(e^2 - 1)$ and we conclude that $y(t) = \frac{1}{2}(e^2 - 1)e^{-2t}$ for $t > 1$.

2.5 Autonomous Equations and Population Dynamics
Practice Problems 2.5: 2, 3, 6, 9, 10

2.(a) Below we sketch the graph of f for $r = 1 = K$.



The critical points occur at $y^* = 0, K$. Since $f'(0) > 0$, $y^* = 0$ is unstable. Since $f'(K) < 0$, $y^* = K$ is asymptotically stable.

(b) We calculate y''. Using the chain rule, we see that

$$y'' = ry'\left[\ln\left(\frac{K}{y}\right) - 1\right].$$

We see that $y'' = 0$ when $y' = 0$ (meaning $y = 0, K$) or when $\ln(K/y) - 1 = 0$, meaning $y = K/e$. Looking at the sign of y'' in the intervals $0 < y < K/e$ and $K/e < y < K$, we conclude that y is concave up in the interval $0 < y < K/e$ and concave down in the interval $K/e < y < K$.

3.(a) Using the substitution $u = \ln(y/K)$ and differentiating both sides with respect to t, we conclude that $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$. The solution of this equation is $u = u_0 e^{-rt}$. Therefore,

$$\frac{y}{K} = \exp[\ln(y_0/K)e^{-rt}].$$

(b) For $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$, we conclude that $y(2) \approx 57.58 \times 10^6$.

(c) Solving the equation in part (a) for t, we see that

$$t = -\frac{1}{r}\ln\left[\frac{\ln(y/K)}{\ln(y_0/K)}\right].$$

Plugging in the given values, we conclude that $\tau \approx 2.21$ years.

6.(a) The equilibrium points are $y^* = 0, 1$. Since $f'(0) = \alpha > 0$, the equilibrium solution $y^* = 0$ is unstable. Since $f'(1) = -\alpha < 0$, the equilibrium solution $y^* = 1$ is asymptotically stable.

(b) The equation is separable. The solution is given by

$$y(t) = \frac{y_0}{e^{-\alpha t} - y_0 e^{-\alpha t} + y_0} = \frac{y_0}{e^{-\alpha t} + y_0(1 - e^{-\alpha t})}.$$

We see that $\lim_{t\to\infty} y(t) = 1$.

9.(a) Since the critical points are $x^* = p, q$, we will look at their stability. Since $f'(x) = -\alpha q - \alpha p + 2\alpha x^2$, we see that $f'(p) = \alpha(2p^2 - q - p)$ and $f'(q) = \alpha(2q^2 - q - p)$. Now if $p > q$, then $-p < -q$, and, therefore, $f'(q) = \alpha(2q^2 - q - p) < \alpha(2q^2 - 2q) = 2\alpha q(q-1) < 0$ since $0 < q < 1$. Therefore, if $p > q$, $f'(q) < 0$, and, therefore, $x^* = q$ is asymptotically stable. Similarly, if $p < q$, then $x^* = p$ is asymptotically stable, and therefore, we can conclude that $x(t) \to \min\{p, q\}$ as $t \to \infty$.
The equation is separable. It can be solved by using partial fractions as follows. We can rewrite the equation as

$$\left(\frac{1/(q-p)}{p-x} + \frac{1/(p-q)}{q-x}\right) dx = \alpha dt,$$

which implies

$$\ln\left|\frac{p-x}{q-x}\right| = (p-q)\alpha t + C.$$

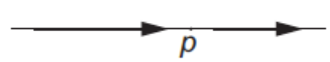
The initial condition $x_0 = 0$ implies $C = \ln|p/q|$, and, therefore,

$$\ln\left|\frac{q(p-x)}{p(q-x)}\right| = (p-q)\alpha t.$$

Applying the exponential function and simplifying, we conclude that

$$x(t) = \frac{pq(e^{(p-q)\alpha t} - 1)}{pe^{(p-q)\alpha t} - q}.$$

(b) In this case, $x^* = p$ is the only critical point. Since $f(x) = \alpha(p-x)^2$ is concave up, we conclude that $x^* = p$ is semistable. Further, if $x_0 = 0$, we can conclude that $x \to p$ as $t \to \infty$. The phase line is shown below.

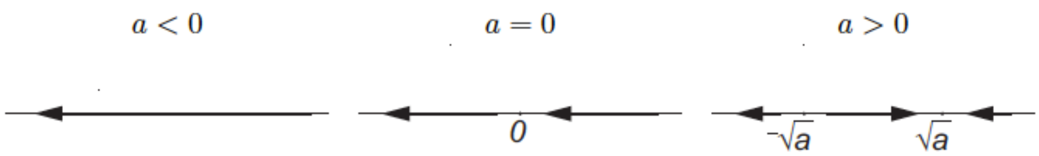


This equation is separable. Its solution is given by

$$x(t) = \frac{p^2 \alpha t}{p\alpha t + 1}.$$

10.(a) The critical points occur when $a - y^2 = 0$. If $a < 0$, there are no critical points. If $a = 0$, then $y^* = 0$ is the only critical point. If $a > 0$, then $y^* = \pm\sqrt{a}$ are the two critical points.

(b) We note that $f'(y) = -2y$. Therefore, $f'(\sqrt{a}) < 0$ which implies that $\sqrt{a}$ is asymptotically stable; $f'(-\sqrt{a}) > 0$ which implies $-\sqrt{a}$ is unstable; the behavior of f' around $y^* = 0$ implies that $y^* = 0$ is semistable. The phase lines are shown below.



(c) Below, we graph solutions in the case $a = -1$, $a = 0$, and $a = 1$, respectively.