MATH 226: Notes on Assignment 5

Practice Problems 2.3: 3, 8, 9, 13, 21

3. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in} \ - \ \text{rate out}.$$

Here, water containing 1/4 lb/gallon of salt is flowing in at a rate of 4 gallons/minute. The salt is flowing out at the rate of $(Q/160)$ lb/gallon $\cdot$ 4 gallons/minute $= (Q/40)$ lb/minute. Therefore,

$$\frac{dQ}{dt} = 1 - \frac{Q}{40}.$$

The solution of this equation is $Q(t) = 40 + Ce^{-t/40}$. Since $Q(0) = 0$ grams, $C = -40$. Therefore, $Q(t) = 40(1 - e^{-t/40})$ for $0 \le t \le 8$ minutes. After 8 minutes, the amount of salt in the tank is $Q(8) = 40(1 - e^{-1/5}) \approx 7.25$ lbs. Starting at that time (and resetting the time variable), the new equation for dQ/dt is given by

$$\frac{dQ}{dt} = -\frac{3Q}{80},$$

since fresh water is being added. The solution of this equation is $Q(t) = Ce^{-3t/80}$. Since we are now starting with 7.25 lbs of salt, $Q(0) = 7.25 = C$. Therefore, $Q(t) = 7.25e^{-3t/80}$. After 8 minutes, $Q(8) = 7.25e^{-3/10} \approx 5.37$ lbs.

8.(a) The differential equation describing the rate of change of cholesterol is $c' = r(c_n - c) + k$, where c_n is the body's natural cholesterol level. Thus $c' = -rc + rc_n + k$; this linear equation can be solved by using the integrating factor $\mu = e^{rt}$. We obtain that $c(t) = k/r + c_n + de^{-rt}$; also, $c(0) = k/r + c_n + d$, thus the integration constant is $d = c(0) - k/r - c_n$. The solution is $c(t) = c_n + k/r + (c(0) - c_n - k/r)e^{-rt}$. If $c(0) = 150$, $r = 0.10$, and $c_n = 100$, we obtain that $c(t) = 100 + 10k + (50 - 10k)e^{-t/10}$. Then $c(10) = 100 + 10k + (50 - 10k)e^{-1}$.

(b) The limit of $c(t)$ as $t \to \infty$ is $c_n + k/r = 100 + 25/0.1 = 350$.

(c) We need that $c_n + k/r = 180$, thus $k = 80r = 8$.

9.(a) The differential equation for the amount of poison in the keg is given by $Q' = 5 \cdot 0.5 - 0.5 \cdot Q/500 = 5/2 - Q/1000$. Then using the initial condition $Q(0) = 0$ and the integrating factor $\mu = e^{t/1000}$ we obtain $Q(t) = 2500 - 2500e^{-t/1000}$.

(b) To reach the concentration 0.005 g/L, the amount $Q(T) = 2500(1 - e^{T/1000}) = 2.5$ g. Thus $T = 1000 \ln(1000/999) \approx 1$ minute.

(c) The estimate is 1 minute, because to pour in 2.5 grams of poison without removing the mixture, we have to pour in a half liter of the liquid containing the poison. This takes 1 minute.

13. (a) Let S(t) be the balance due in dollars on the loan at time t months. The monthly interest rate is then .09/12 so the differential equation for S is

$$\frac{dS}{dt} = \frac{.09}{12}S - 800\left(1 + \frac{t}{120}\right).$$

This is a linear differential equation. Its solution is

$$S(t) = \frac{1111111111}{1250000}t + \frac{2111111111}{9375} + C\,e^{\frac{3}{400}t}$$

or, in decimal format,

$$S(t) = 888.89 + 225{,}185.19 + C\,e^{.0075t}$$

The initial condition is S(0) = 100000 which implies that C =-1173611111/9375 = -125185.19. The particular solution then is

$$S(t) = 888.89 + 225.185.19 - 125{,}185.19\,e^{.0075t}$$

To find when the loan will be paid, we need to solve *S(T)* = 0 for *T*. You can do this by having Maple plot the graph of *S(t)* and see where it intersects the horizontal axis or use the *fsolve* command to obtain 135.36 months, which is about 11.28 years.

(b) We have the same differential equation but our "initial" condition is S(240) = 0 since we want the loan paid off in precisely 20 years = 240 months. This condition makes C = -72486.62356 so the particular solution is

$$S(t) = 888.89 + 225.185.19 - 72486.62e^{.0075t}$$

Therefore *S(0)* = 152,698.56, so the initial loan could be as high as $152,698.56.

21.(a) The differential equation for Q is

$$\frac{dQ}{dt} = kr + P - \frac{Q(t)}{V}r.$$

Therefore,

$$V\frac{dc}{dt} = kr + P - c(t)r.$$

The solution of this equation is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. Therefore $\lim_{t\to\infty} c(t) = k + P/r$.

(b) In this case, we will have $c(t) = c_0 e^{-rt/V}$. The reduction times are $T_{50} = \ln(2)V/r$ and $T_{10} = \ln(10)V/r$.

(c) Using the results from part (b), we have: Superior, $T = 430.85$ years; Michigan, $T = 71.4$ years; Erie, $T = 6.05$ years; Ontario, $T = 17.6$ years.