MATH 226

Notes on Assignment 10

3.3 Homogeneous Linear Systems with Constant Coefficients

Practice Problems: 1*, 5*, 6*, 10*, 15*, 17, 18, 34

1. We look for eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

We see that $\det(A - \lambda I) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. Therefore, the eigenvalues are given by $\lambda = 2, -1$. First, $\lambda_1 = 2$ implies

$$A - \lambda_1 I = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an eigenvector associated with λ_1 and

$$\mathbf{x}_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is one solution of the system. Second, $\lambda_2 = -1$ implies

$$A - \lambda_2 I = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}.$$

Therefore,

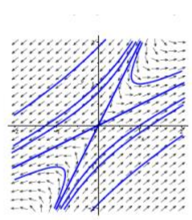
$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is an eigenvector associated with λ_2 and

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a second solution of the system. Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$



If the initial condition is a multiple of $\begin{pmatrix} 1 & 2 \end{pmatrix}^T$, then the solution will tend to the origin along the eigenvector $\begin{pmatrix} 1 & 2 \end{pmatrix}^T$. Otherwise, the solution will grow, following the eigenvector $\begin{pmatrix} 2 & 1 \end{pmatrix}^T$.

5. We look for eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}.$$

We see that $\det(A - \lambda I) = \lambda^2 + 2\lambda = \lambda(\lambda + 2)$. Therefore, the eigenvalues are given by $\lambda = 0, -2$. First, $\lambda_1 = 0$ implies

$$A - \lambda_1 I = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

is an eigenvector associated with λ_1 and

$$\mathbf{x}_1(t) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

is one solution of the system. Second, $\lambda_2 = -2$ implies

$$A - \lambda_2 I = \begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix}.$$

Therefore,

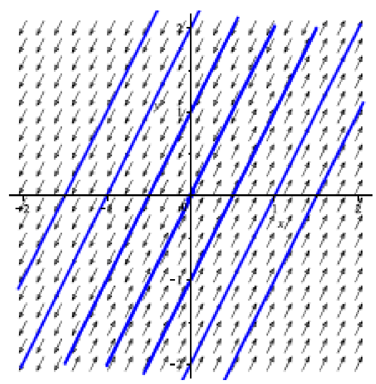
$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is an eigenvector associated with λ_2 and

$$\mathbf{x}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a second solution of the system. Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$



Any solution starting at a point on the line of critical points remains fixed for all time at its starting point. A solution starting at any other point in the plane moves on a line parallel to $\mathbf{v}_2$ toward the point of intersection of this line with the line of critical points.

6. We look for eigenvalues and eigenvectors of

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

We see that $\det(A - \lambda I) = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3)$. Therefore, the eigenvalues are given by $\lambda = -1, -3$. First, $\lambda_1 = -1$ implies

$$A - \lambda_1 I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector associated with λ_1 and

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is one solution of the system. Second, $\lambda_2 = -3$ implies

$$A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Therefore,

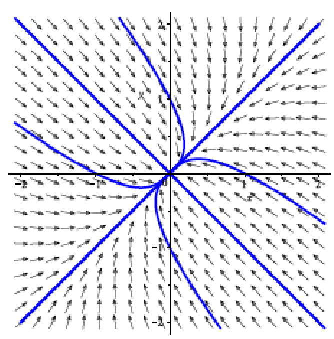
$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector associated with λ_2 and

$$\mathbf{x}_2(t) = e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a second solution of the system. Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$



If the initial condition is a multiple of $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$, then the solution will tend to the origin along the eigenvector $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$. Otherwise, the solution will tend to the origin, following the eigenvector $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

10. We look for eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}.$$

We see that $\det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$. Therefore, the eigenvalues are given by $\lambda = 4, 2$. First, $\lambda_1 = 4$ implies

$$A - \lambda_1 I = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}.$$

Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector associated with λ_1 and

$$\mathbf{x}_1(t) = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is one solution of the system. Second, $\lambda_2 = 2$ implies

$$A - \lambda_2 I = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}.$$

Therefore,

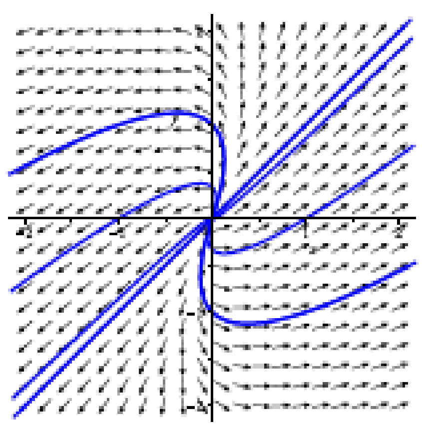
$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

is an eigenvector associated with λ_2 and

$$\mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

is a second solution of the system. Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$



If the initial condition is a multiple of $\begin{pmatrix} 1 & 3 \end{pmatrix}^T$, then the solution will grow, staying on the eigenvector $\begin{pmatrix} 1 & 3 \end{pmatrix}^T$. Otherwise, the solution will grow, following the eigenvector $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

15. From problem 10, the general solution is

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ implies

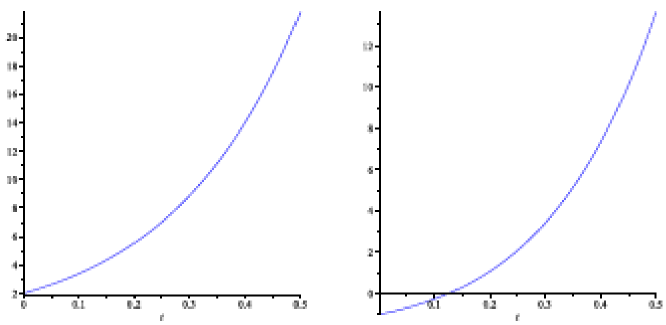
$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 7/2 \end{pmatrix}.$$

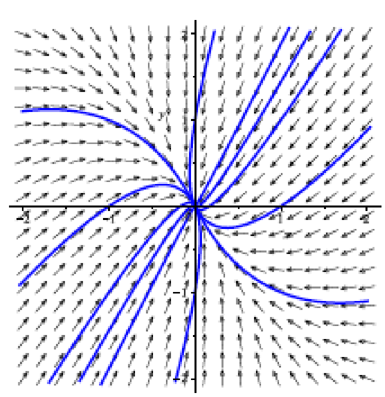
Therefore, the solution is

$$\mathbf{x}(t) = -\frac{3}{2}e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{7}{2}e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

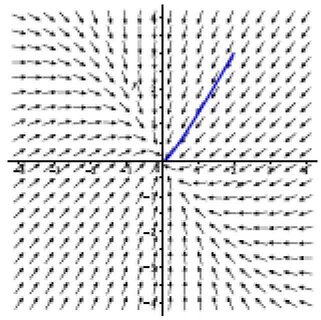


Both components tend to $+\infty$ as $t \to \infty$.

17.(a)



(b)



(c) The general solution is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

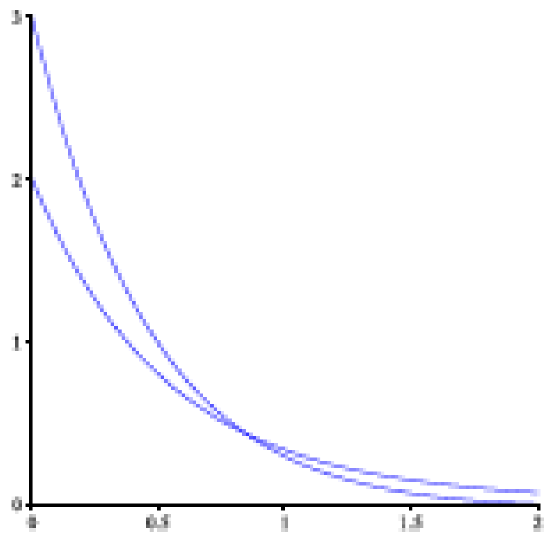
The initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ implies

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 7/4 \end{pmatrix}.$$

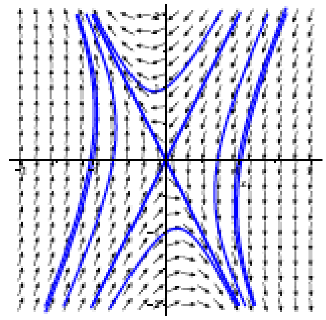
Therefore, the solution passing through the initial point $(2,3)$ is given by

$$\mathbf{x}(t) = -\frac{1}{4} e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{7}{4} e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

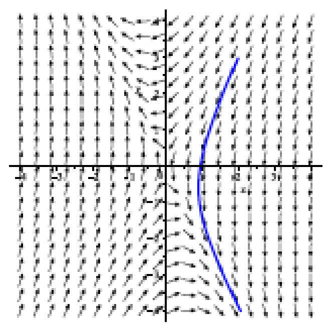
The component plots are shown below.



18.(a)



(b)



(c) The general solution is given by

$$\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

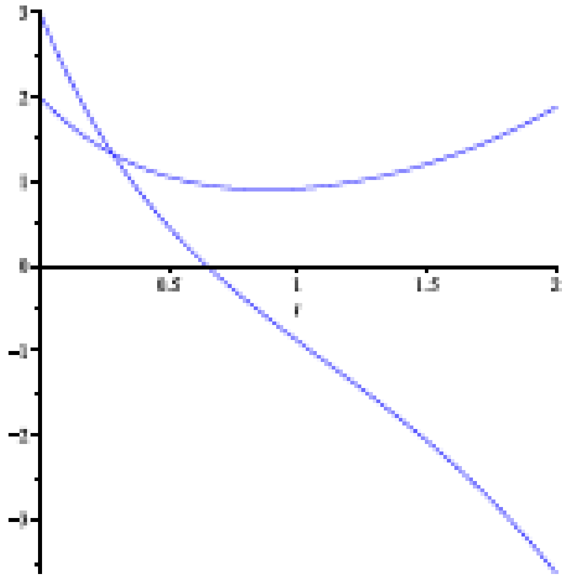
The initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ implies

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 7/4 \end{pmatrix}.$$

Therefore, the solution passing through the initial point $(2, 3)$ is given by

$$\mathbf{x}(t) = -\frac{1}{4} e^{t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{7}{4} e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The component plots are shown below.



34.(a) For $\alpha = 0.5$, the characteristic equation is $2\lambda^2+4\lambda+1 = 0$. Therefore, the eigenvalues are $\lambda = -1 \pm 1/\sqrt{2}$. The eigenvector corresponding to $\lambda_1 = -1+1/\sqrt{2}$ is $\mathbf{v}_1 = \begin{pmatrix} -\sqrt{2} & 1 \end{pmatrix}^T$. The eigenvector corresponding to $\lambda_2 = -1-1/\sqrt{2}$ is $\mathbf{v}_2 = \begin{pmatrix} \sqrt{2} & 1 \end{pmatrix}^T$. Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^{(-1+1/\sqrt{2})t} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + c_2 e^{(-1-1/\sqrt{2})t} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}.$$

Since both eigenvalues are negative, the equilibrium point is a stable node.

(b) For $\alpha = 2$, the characteristic equation is $\lambda^2 + 2\lambda - 1 = 0$. Therefore, the eigenvalues are $\lambda = -1 \pm \sqrt{2}$. The eigenvector corresponding to $\lambda_1 = -1 + \sqrt{2}$ is $\mathbf{v}_1 = \begin{pmatrix} 1 & -\sqrt{2} \end{pmatrix}^T$. The eigenvector corresponding to $\lambda_2 = -1 - \sqrt{2}$ is $\mathbf{v}_2 = \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}^T$. Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^{(-1+\sqrt{2})t} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} + c_2 e^{(-1-\sqrt{2})t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

Since the eigenvalues have opposite sign, the equilibrium point is a saddle point.

(c) For general α, the characteristic equation is $\lambda^2 + 2\lambda + 1 - \alpha = 0$. The eigenvalues are given by $\lambda = -1 \pm \sqrt{\alpha}$. For $0.5 \leq \alpha \leq 2$, both eigenvalues are real and clearly $-1 - \sqrt{\alpha} < 0$. Therefore, we just need to determine whether $-1 + \sqrt{\alpha} = 0$. We see this occurs when $\alpha = 1$. At that value the equilibrium point switches from a saddle point (for $\alpha > 1$) to a stable node (for $\alpha < 1$).