MATH 226 *Differential Equations*
Notes on Assignment 1

Problem 1: $\int xe^{2x^2}dx$
Let $u = 2x^2$ so $du = 4xdx$ and $xdx = \frac{1}{4}du$ and $e^{2x^2} = e^u$.
Hence $\int xe^{2x^2}dx = \int \frac{1}{4}e^u du = \frac{1}{4}e^u + C = \frac{1}{4}e^{2x^2} + C$

Problem 3: $\int t^2\cos(1-4t^3)dt$
Let $u = 1-4t^3$ so $du = -12t^2st$ and $t^2dt = -\frac{1}{12}du$.
Hence $\int t^2\cos(1-4t^3)dt = \int -\frac{1}{12}\cos udu = -\frac{1}{12}\sin u + C = -\frac{1}{12}\sin(1-4t^3) + C$

Problem 4 : $\int x^2\sin(2x)dx$
Use Integration By Parts with $U = x^2, dV = \sin(2x)dx$. Then $dU = 2xdx, V = -\frac{1}{2}\cos(2x)$ so
$UV - \int VdU = -\frac{1}{2}x^2\cos(2x) - \int -x\cos(2x)dx = -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)dx$.
Use Integration By Parts on $\int x\cos(2x)dx$ with $U = x, dV = \cos(2x)d$ giving $dU = dx, V = \frac{1}{2}\sin(2x)$ which yields $\int x\cos(2x)dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin(2x)dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos(2x) + C$.
Finally, $\int x^2\sin(2x)dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C$.

Problem 5 : $\int_1^e \cos(\ln x)dx$
First analyze the indefinite integral using Integration By Parts twice.
(a) Let $U = \cos(\ln x)), dV = 1dx$. Then $dU = \frac{-\sin(\ln x)}{x}dx, V = x$
so $UV = x\cos(\ln x), VdU = x\frac{-\sin(\ln x)}{x}dx = -\sin(\ln x)dx$
Thus $\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$
(b) Use Integration By Parts on $\int \sin(\ln x)dx$ with $U = \sin(\ln x), dV = 1dx$ so $dU = \frac{\cos(\ln x)}{x}dx, V = x$
so $\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)$
Putting (a) and (b) together, we have $\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx$ so
$2\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) = x[\cos(\ln x) + \sin(\ln x)]$ and
$\int \cos(\ln x)dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)]$
Hence the value of the definite integral is $\int_1^e \cos(\ln x)dx = \frac{e}{2}[\cos(\ln e) + \sin(\ln(e)] - \frac{1}{2}[\cos(\ln 1) + \sin(\ln 1) = \frac{e}{2}[\cos 1 + \sin 1] - \frac{1}{2}[\cos 0 + sin0] = \frac{e}{2}[\cos 1 + \sin 1] - \frac{1}{2}$ since $\ln e = 1, \ln 1 = 0, \cos 0 = 1, \sin 0 = 0$

Problem 6: $\int \sin^4 x\cos^3 xdx$
Note $\sin^4 x\cos^3 x = \sin^4 x\cos^2 x\cos x = \sin^4 x(1-\sin^2 x)\cos x = (\sin^4 x - sin^6 x)\cos x$
Let $u = \sin x$ so $du = \cos xdx$. Then $\int \sin^4 x\cos^3 xdx = \int (u^4 - u^6)du = \frac{u^5}{5} = \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

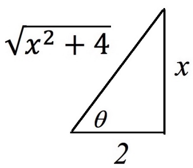
Problem 7: $\int \cot 5xdx$
Let $u = \sin(5x)$ so $\frac{1}{5}du = \cos(5x)$. Then $\int \cot(5x)dx = \int \frac{\cos(5x)}{\sin(5x)}dx = \int \frac{1}{5}\frac{1}{u}du = \frac{1}{5}|\ln u| + C = \frac{1}{5}|\ln(\sin(5x)| + C$

Problem 10: $\int_1^2 x^{2/5}\ln xdx$
Use Integration By Parts with $U = \ln x, dV = x^{2/5}dx$.
Then $dU = \frac{1}{x}dx, V = \frac{5}{7}x^{7/5}$ so $UV = \frac{5}{7}x^{7/5}\ln x, VdU = \frac{5}{7}\frac{x^{7/5}}{x} = \frac{5}{7}x^{2/5}$
Now $\int x^{2/5}\ln xdx = \frac{5}{7}x^{7/5}\ln x - \int \frac{5}{7}x^{2/5}dx = \frac{5}{7}x^{7/5}\ln x - \frac{5}{7}\frac{5}{7}x^{7/5} + C = \frac{5}{7}x^{7/5}\ln x - \frac{25}{49}x^{7/5} + C$
Evaluating at $x = 2$ and $x = 1$ gives $\int_1^2 x^{2/5}\ln xdx = \frac{5}{7}2^{7/5}\ln 2 - \frac{25}{49}2^{7/5} + \frac{25}{49} = \frac{10}{7}x^{2/5}\ln 2 - \frac{5}{49}2^{2/5} + \frac{25}{49}$

Problem 11: $\int \frac{x+1}{x^2-4} dx$
Use Partial Fraction Decomposition: $\frac{x+1}{x^2-4} = \frac{x+1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2)+B(x+2)}{x^2-4} = \frac{(A+B)x-2A+2B}{x^2-4}$
Equating numerators yields $A + B = 1, -2A + 2B = 1$ so $A = \frac{1}{4}, B = \frac{3}{4}$.
Hence

$$\int \frac{x+1}{x^2-4} dx = \int \frac{\frac{1}{4}}{x+2} + \frac{\frac{3}{4}}{x-2} dx = \frac{1}{4} \ln|x+2| + \frac{3}{4} \ln|x-2| + C$$

Problem 13: $\int \frac{1}{x(x^2+4)} dx$
Here is an alternative solution using a trigonometric substitution based on this right triangle:



$\tan\theta = x/2$ so let $x = 2\tan\theta$ and $dx = 2\sec^2\theta d\theta$.
From the triangle, $\sec\theta = \frac{\sqrt{x^2+4}}{2}$ so $\sqrt{x^2+4} = 2\sec\theta$ giving $x^2 + 4 = 4\sec^2\theta$.
Then $\int \frac{1}{x(x^2+4} dx = \int \frac{2\sec^2\theta d\theta}{(2\tan\theta)(4\sec^2\theta)} = \int \frac{1}{4}\cot\theta d\theta = \frac{1}{4}\ln|\sin\theta| + C = \frac{1}{4}\ln|\frac{x}{\sqrt{x^2+4}}| + C$

Problem 14: $\int sec^2(\pi x) dx$
Let $u = \pi x$ so $du = \pi dx$ and $dx = \frac{1}{\pi} du$. Then $\int sec^2(\pi x) dx = \frac{1}{\pi} \int sec^2(u) du = \frac{1}{\pi}\tan u + C = \frac{1}{\pi}\tan(\pi x) + C$