# Project 3

**Objectives:**

- Gain practice using numerical methods to approximate solutions of differential equations
- Understand the importance and limitations of numerical approximations to solutions of differential equations
- Learn how to effectively communicate your work through a written report

**Instructions:**
You and your teammates will answer the following questions in a MATLAB live script. Your responses to each question should include complete sentences, easy-to-follow mathematics, and discussion whenever prompted. Each question asks for a ***summary paragraph*** at the end to relay what you've learned. All graphs should include titles, labels and captions. You will ***submit one pdf of the report on Canvas, in addition to your mlx file, per pair***.

**Some advice and expectations:**

- The target audience for your written report is a differential equations student with some basic knowledge of MATLAB and minimal knowledge of numerical methods.
- Your MATLAB code should contain no errors. Proofread your mlx file carefully before you submit. Make sure each problem is clearly labeled.
- Please include semi-colons after each output that you do not wish to talk about, and make sure all other output appear ***below*** the code segments of your file.
- Your discussion answers must use correct spelling, grammar, and syntax.
- Don't forget to label the title and axes of all graphs. Use `title('here is a ... title')`, `xlabel('x label here')`, and `ylabel('y label here')`.
- Annotate your code portions with concise `% green comments` explaining what you are doing.
- All portions of this assignment should be a joint effort from you and your teammates.
- Make sure you type and sign the honor code pledge at the end of the report.
- Utilize office hours! I'm here to help.

*****You are strongly encouraged to start this project as soon as possible. It is not meant to be completed in one sitting!***

1. Consider the following initial value problem

$$\frac{dy}{dt} = -2ty^2 \quad y(0) = 1 \tag{1}$$

   (a) Solve this IVP (Initial Value Problem) by hand.

   (b) Write out the approximation, $\hat{y}_{n+1}$, to this differential equation using Euler's method. In your own words, describe how Euler's method works.

   (c) Write out the approximation, $\hat{y}_{n+1}$, to this differential equation using Improved Euler's method. In your own words, describe how Improved Euler's method works.

   (d) Download the following codes from the *Project 3 Files* in the Handouts folder: `myEuler.m`, `myImprovedEuler.m`, and `myRK4.m`. You will also see a live script that illustrates how to use these functions.

   (e) Compute the approximation of the solution to Eq (1) using all three methods for the interval $t \in [0, 2]$ with step size $h = 0.1$. Plot the exact solution as a curve, together with each approximation as different markers. Do not connect the markers with a line. Comment on the accuracy of each approximation as determined from the plot. [1]

   (f) Calculate the global error for each method:

$$E_n = \max(\text{abs}(y(t_n) - \hat{y}_n))$$

   where $y(t_n)$ is the exact solution from (a) and $\hat{y}_n$ is the approximation from each of the schemes. Does the error support your intuition from the plot in (e)?

   (g) Compute new approximations using a smaller step size of $h/2 = 0.05$ (note that the number of steps $n$ will need to increase) and compute the error for each method using this new step size. Use the following formula to compute the order of convergence, $p$, for each method

$$p \approx \log_2\left(\frac{E_n(h)}{E_{2n}(h/2)}\right),$$

   where $E_n(h)$ is the global error corresponding to the approximation of step size $h$ and $E_{2n}(h/2)$ is the error association with the approximation of step size $h/2$.

   (h) Compute the approximations for four more step sizes, halving the step size each time. Plot the error vs the step size, $h$, on a loglog plot (try `loglog` in MATLAB). Where do you see the order of convergence $p$ showing up on this plot? Why?

   (i) Write a paragraph summarizing your results from this problem. When would someone want to use Euler's method vs. the other methods? When does Euler's method not do a good job? What are some of the potential down sides to using the other methods?

[1] It may be difficult to see your markers on the plot. Feel free to use `'Linewidth',2` to make the edge of the markers thicker and `'MarkerSize',8` to make the marker larger.

2. Consider the following initial value problem

$$\frac{dy}{dt} = \frac{3t^2}{3y^2 - 2}, \quad y(0) = 0 \tag{2}$$

(a) Solve this differential equation by hand to find an implicit function involving the solution $y(t)$ to Eq (2).

(b) Use `quiver` to plot the direction field on the grid `[-3:0.2:3]`×`[-3:0.2:3]` and `fimplicit` to plot your solution from (a) on top of the direction field in MATLAB.

(c) We want to numerically approximate the solution to Eq (2) from $t = 0$ to some endpoint $t = b > 0$. Use the direction field to estimate the value of $b$ for which your numerical approximation will start to fail. Why does it fail there? How can you find this value from the differential equation?

(d) Use `myEuler.m` and `myRK4.m` to find the approximate solution to the differential equation from $t = 0$ to $t = 2$ using a step size of $h = 0.01$. Plot the approximations on top of the direction field using two different markers (e.g., circles and x's). Do not connect the markers with lines and make sure to include a legend indicating which marker is which approximation.

(e) Even though we've plotted past the end point you should have found in (c), the approximation still gives some values. What do you think those values mean?

(f) Repeat the analysis for the integral curve going through $(0, 1)$. For how long is this approximation valid? Draw the direction field and corresponding integral curve, as well as the approximations using both Euler and RK4 methods.

(g) Write a summary paragraph describing numerical methods for nonlinear equations. Why does the numerical approximation fail? How could we have known this from the outset? What is the take-home message?

3. We can derive Euler's Method from considering the Taylor series of the solution $y(t)$ about $t = t_n$

$$y(t) = y(t_n) + y'(t_n)(t - t_n) + \frac{y''}{2}(t - t_n)^2 + \dots$$

and truncating it after the linear term to yield an approximation for $y(t_{n+1})$:

$$\hat{y}_{n+1} = \hat{y}_n + hf(t_n, \hat{y}_n).$$

One can imagine that if you keep higher-order terms, one could do a better job approximating the solution.

(a) Truncate the Taylor series for $y(t_{n+1})$ after the quadratic term to yield the second-order Taylor Method. Note that you will need to write $y''(t)$ in terms of the only known information, which is the function $f(t, y)$ from the differential equation $y'(t) = f(t, y)$.

(b) Write the second-order Taylor method approximation for the differential equation in Eq (1), re-written here

$$\frac{dy}{dt} = -2ty^2$$

(c) Write a MATLAB function that will compute the second-order Taylor method for the IVP in Eq (1). Use your function to compute the approximation and plot it against the real solution. Then, show that the method is 2nd order convergent.

(d) Reflect about why we might prefer to use the Improved Euler method over this one? Why don't we just include lots of higher-order terms in the approximation to make it as accurate as we want?