MATH 226 Spring 2022

Team Project 1 (Corrected)

Part I

1. We say that a population P *grows at a constant per capita rate* if at each instant $\frac{P'}{P}$ has the same constant value r. More carefully, we should write $\frac{P'(t)}{P(t)} = r$ so P satisfies the differential equation $P' = rP$ and hence $P(t) = P_0 e^{rt}$ where P_0=P(0).

Thus a constant per capita growth rate leads to exponential growth (if $r > 0$) or exponential decay (if $r < 0$).

If a population grows this way and we know the population at time 0 and at any single later time ($P_1 = P(t_1)$ for some $t_1 > 0$), show how to determine r.

2. Below is a table showing United States census data (in millions of people) from 1790 to 2020. The census attempts to count the people living in the U. S. on April 1 of that year.
Let $t = 0$ correspond to the year 1790 of the initial census.

(a) Estimate r using census data for the years 1790 and 1800.

(b) Estimate r using census data for the years 1790 and 1810.

(c) Estimate r using census data for the years 1790 and some other year between 1820 and 1850.

T	*Year*	*Population*	*t*	*Year*	*Population*	*t*	*Year*	*Population*
0	1790	3.9292	**80**	1870	38.5584	**160**	1950	151.3258
10	1800	5.3085	**90**	1880	50.1892	**170**	1960	179.3232
20	1810	7.2399	**100**	1890	62.9798	**180**	1970	203.2119
30	1820	9.6385	**110**	1900	76.2122	**190**	1980	226.5458
40	1830	12.8660	**120**	1910	92.2285	**200**	1990	248.7099
50	1840	17.0695	**130**	1920	106.0215	**210**	2000	281.4219
60	1850	23.1919	**140**	1930	123.2026	**220**	2010	308.7455
70	1860	31.4433	**150**	1940	132.1646	**230**	2020	331.4493

3. Use the r values you found in Problem 2 to predict the U. S. Population in 1860. Which r value gives the best prediction?

4. When Thomas Jefferson became President in 1801, he knew the results of the 1790 and the 1800 censuses. Using this data and assuming an exponential growth model, Jefferson could have estimated the U.S. population for each of the next 60 years. How accurate would his predictions have been? [You might examine (a) the actual differences between the predicted and observed values and (b) the relative difference: the absolute difference divided by the observed values.]

5. Compare one r against another one using *Sum of Squared Errors*. The Sum of Squares method is a measure of how well a particular predictive model fits observed data. Suppose we have a set of n times $t_1, t_2,.., t_i,..,t_n$ at each of which we have an observed value y_i and a value x_i predicted by a model that has r as a parameter/ . [In our case, the observed values are the census data and the predicted populations are given by our function $x_i = P(t_i) = P_0 e^{rt_i}$]. The Sum of Squares method adds up the squares of the differences between y_i and x_i: $\sum_{i=1}^{n}(y_i - x_i)^2$ We would

say that "r_1 is a better value than r_2" if the Sum of Squared Errors associated with r_1 is smaller than the Sum of Squared Errors for r_2.

Find the Sum of Squared Errors for each of the three *r* values you found in Question 2. Which the best one?

6. We would like to find the best possible *r* value of the infinitely many possible choices we might make; that is, we would like to **minimize** the Sum of Squares. This is called the *Method of Least Squares*. Solving for the minimum is in general a hard problem for arbitrary predictive model. Fortunately, we can find an exact solution if the predictive mode is linear. The *Method of Least Squares* finds the "best" straight line $f(x) = ax + b$ through a set of data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by choosing a and b to minimize $\sum_{i=1}^{n}(y_i - (ax_i + b))^2$ which is the sum of the squares of the vertical distances between the data and the line. The solution (which is not hard to derive if you have taken Multivariable Calculus) is

$$a = \frac{\sum_{i=1}^{n}(x_i - x^*)(y_i - y^*)}{\sum_{i=1}^{n}(x_i - x^*)^2}, \qquad b = y^* - ax^*$$

$$\text{where } x^* = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } y^* = \frac{1}{n}\sum_{i=1}^{n} y_i$$

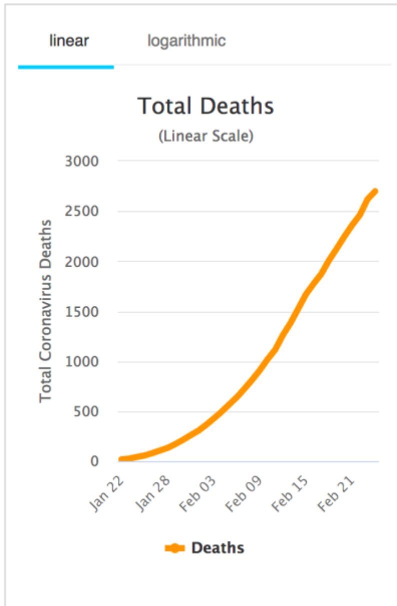
Our exponential growth model $P(t) = P_0 e^{rt}$ I is not a linear one, but if we take the natural logarithm of each side, we do get a linear relationship: $\ln P = \ln P_0 + rt$ which has the form $ax + b$ with $b = \ln P_0$, $a = r$, and $x = t$.

(a) Graph the natural logarithm of the census vs time for the period 1790 – 1860. Does it appear to be approximately a straight line? Use the Method of Least Squares to find the best linear approximation. What is the estimated *r*? Display the graph of the resulting exponential function together with the graph of the census data.

(b) Repeat (a) for the period 1790 – 2010. What does your resulting model predict U.S. population will be on April 1 of this year? On April 1, 2050? On April 1, 2100? Are these plausible predictions?

Part II

The website https://www.worldometers.info/coronavirus/ shows a plot of the total deaths from the coronavirus since the first deaths were reported a month ago.



Here are the daily statistics:

Date	**Total Deaths**
January 23	**25**
January 24	**41**
January 25	**56**
January 26	**80**
January 27	**106**
January 28	**132**
January 29	**170**
January 30	**213**
January 31	**259**
February 1	**304**
February 2	**362**
February 3	**426**
February 4	**492**
February 5	**565**
February 6	**638**
February 7	**724**
February 8	**805**

Date	**Total Deaths**
February 9	**910**
February 10	**1018**
February 11	**1115**
February 12	**1261**
February 13	**1383**
February 14	**1526**
February 15	**1669**
February 16	**1775**
February 17	**1873**
February 18	**2009**
February 19	**2126**
February 20	**2247**
February 21	**2360**
February 22	**2460**
February 23	**2618**
February 24	**2699**

7. Assuming that deaths from this coronavirus are explained by an exponential model, what is your best estimate for r? Use your choice of r to predict the death totals for each of the next 10 days. What does your model predict will be the total deaths by the Ides of March (March 15?)

Part III

8. A possible generalization of the constant per capita growth model is the differential equation $P' = r\,P^k$ where k is some nonzero constant. (k =1 is the classic case). Suppose $r > 0$ and $P(0) > 0$. Without solving the differential equation,
(a) show that P will always be an increasing function of t,
(b) determine for which k is the graph of P as a function of t concave up and for which k is it concave down? You might want to consider the separate cases: $k > 1$, $0 < k < 1$, $k < 0$,
(c) For which values of k does $P(t)$ grow unboundedly? For which values of k does $P(t)$ approach a finite limit asymptotically?

9. Graph some solutions for $r = .03$ and $P(0) = 100$ and different choices of k.

10. Solve the differential equation in Problem 8 for the case $k = -1$.

11. Solve the differential equation in Problem 8 for the case $k = ½$.

12. Solve the differential equation in Problem 8 for the case $k = 2$.

13. Solve the differential equation in Problem 8 for the general case $k \neq 1$.

14. Show that for all $k > 1$, there is a finite time T at which the population would become infinite. Find T in terms of r and Po.

15. What k value best models U. S. population growth between 1790 and 2010?

16. What k value best models deaths from the coronavirus?