MATH 226 :Notes on MATLAB Worksheet 2

Phase Plane Tutorial

This tutorial will walk you through how to use the function plotPhasePlane.m function.

The function plotPhasePlane.m will draw the phase portrait for a 2x2 systems of linear, constant-coefficient, homogeneous differential equations, like

$$\begin{bmatrix} \frac{\mathrm{dy_1}}{\mathrm{dt}} \\ \frac{\mathrm{dy_2}}{\mathrm{dt}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The function has inputs:

A - the 2x2 matrix of coefficients

yBounds - a vector with two entries holding the minimum and maximum values of y for the state space

tBounds - a vector with two entries holding the minimum and maximum values of t for plotting trajectories (you may want to include negative t)

M - the number of trajectories you want to plot

The output of plotPhasePlane.m is a phase portrait that has the straight line solutions given by the eigenvalues as black dashed lines and the trajectores as colored curves. Note that it will not plot any eigenvectors when eigenvalues are complex (i.e., there are no straight line solutions)

```
clear all
```

Richardson Stable Arms Race

```
  % define the matrixx
A = [-5 4;3 -4]
```

```
A = 2x2
    -5     4
     3    -4
```

```
RHS = [1;2];
B = rref([A -RHS]);
eqSoln = B(:,end)
```

```
eqSoln = 2x1
    1.5000
    1.6250
```

```
% find the eigenvalues and eigenvectors of the matrix
[X D] = eig(sym(A))
```

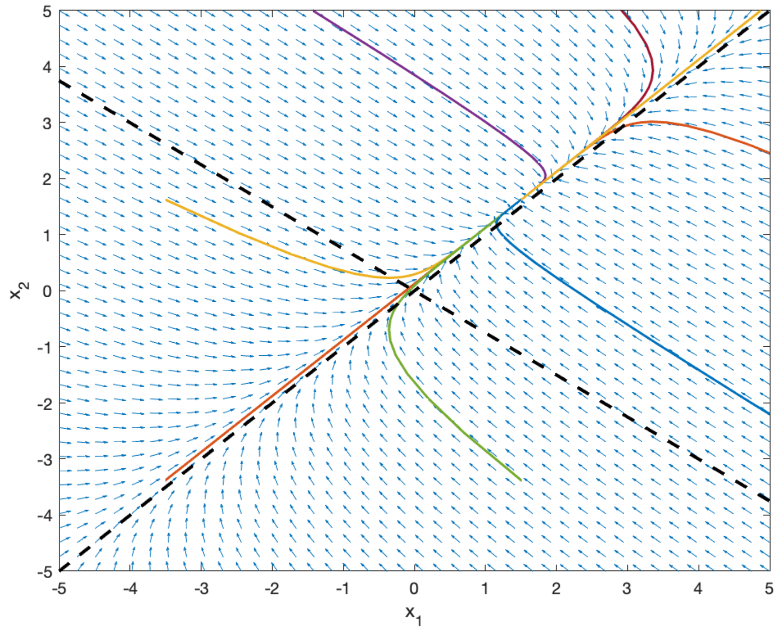
X =

$$\begin{pmatrix} -\frac{4}{3} & 1 \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -8 & 0 \\ 0 & -1 \end{pmatrix}$$

```
% state the bounds for y and t
yBounds = [-5 5];  % [ymin ymax]
tBounds = [0, 50]; % [tmin tmax]
M = 3; % number of trajectories to plot
plotPhasePlane(A, yBounds, tBounds,M,RHS,eqSoln)
```



Note that solutions can be written as

$$Ae^{-t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{-8t}\begin{pmatrix}-4\\3\end{pmatrix} = e^{-t}\left[A\begin{pmatrix}1\\1\end{pmatrix} + Be^{-7t}\begin{pmatrix}-4\\3\end{pmatrix}\right]$$

and the term in square brackets rapidly approaches the eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$.

Richardson Unstable Arms Race

```
A = [-9 11; 12 -8 ]
```

```
A = 2x2
    -9    11
    12    -8
```

```
RHS = [1;2];
B = rref([A -RHS]);
eqSoln = B(:,end)
```

```
eqSoln = 2x1
   -0.5000
   -0.5000
```

```
[X D] = eig(sym(A))
```

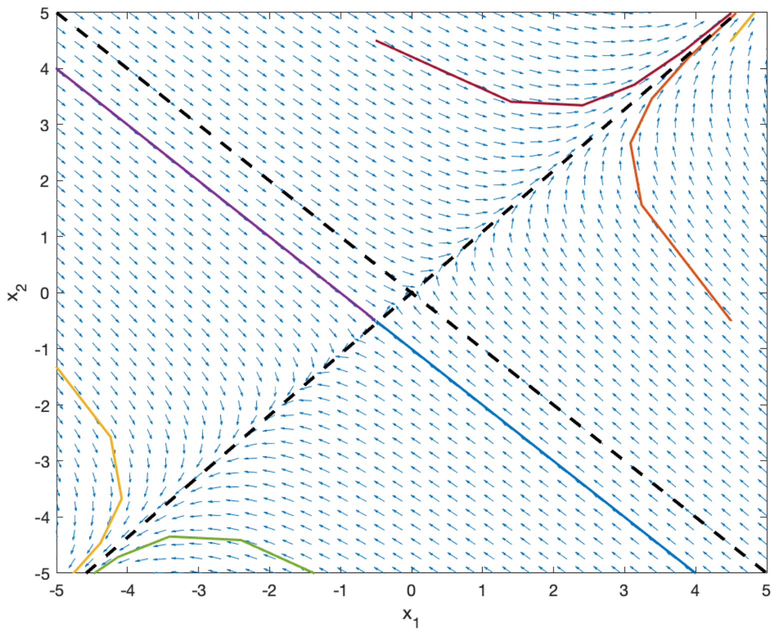
X =

$$\begin{pmatrix} -1 & \frac{11}{12} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -20 & 0 \\ 0 & 3 \end{pmatrix}$$

```
plotPhasePlane(A, yBounds, tBounds,M,RHS,eqSoln)
```



We can write solutions in the form

$$e^{3t}\left[A\begin{pmatrix}11\\12\end{pmatrix} + Be^{-23t}\begin{pmatrix}-1\\1\end{pmatrix}\right]$$

If $A \neq 0$, then the second term in the square brackets rapidly goes to 0 so solutions approach the eigenvector $\begin{pmatrix}11\\12\end{pmatrix}$ as $t \to \infty$. They go out in the positive direction if $A > 0$ and in the negative direction if $A < 0$. In the case $A = 0$, the solutions approach the origin along the other eigenvector.

Zero Eigenvalue Example

```
A = [-3 6; 4 -8]
```

```
A = 2x2
    -3     6
     4    -8
```

```
RHS = [1;2];
B = rref([A -RHS]);
eqSoln = B(:,end)
```

```
eqSoln = 2x1
     0
     1
```

```
[X D] = eig(sym(A))
```

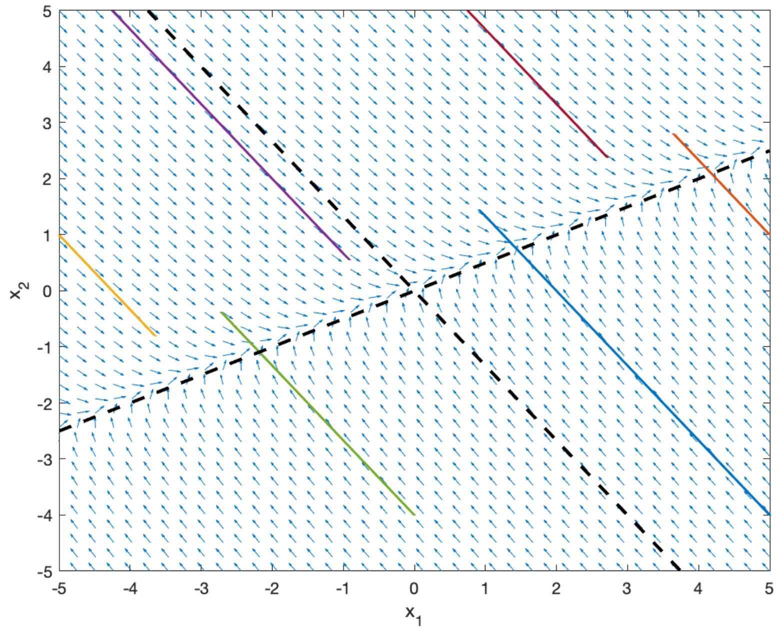
X =

$$\begin{pmatrix} 2 & -\frac{3}{4} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} 0 & 0 \\ 0 & -11 \end{pmatrix}$$

```
plotPhasePlane(A, yBounds, tBounds,M,RHS,eqSoln)
```



Here solutions have the form

$$A\begin{pmatrix}2\\1\end{pmatrix}+Be^{-11t}\begin{pmatrix}-3n\\4\end{pmatrix}$$

The second term rapidly goes to 0 so we approach some point on the eigenvector $\begin{pmatrix}2\\1\end{pmatrix}$.

A Complex Eigennvalue Exampe

```
A = [0 1; -2 0];
RHS = [1;2];
B = rref([A -RHS]);
eqSoln = B(:,end)
```

```
eqSoln = 2x1
     1
    -1
```

```
[X D] = eig(sym(A))
```

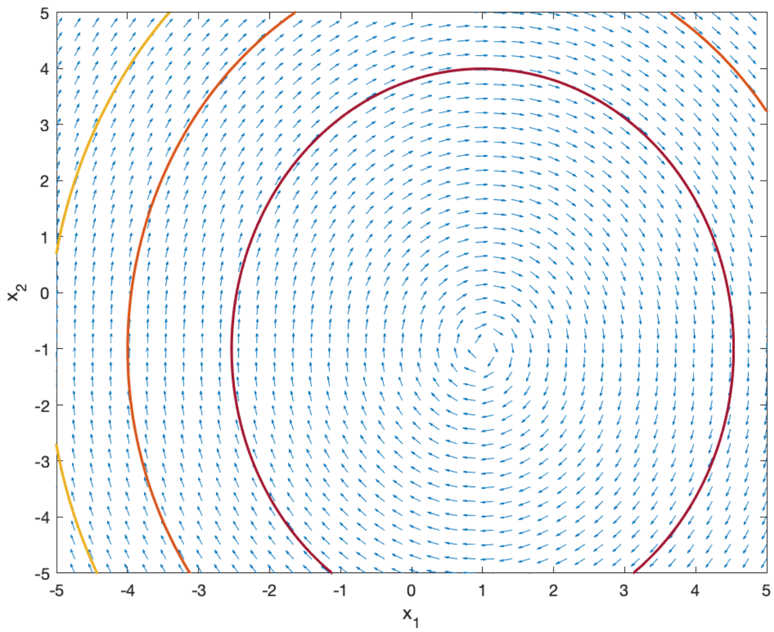
X =

$$\begin{pmatrix} \frac{\sqrt{2}\,i}{2} & -\frac{\sqrt{2}\,i}{2} \\ 1 & 1 \end{pmatrix}$$

D =

$$\begin{pmatrix} -\sqrt{2}\,i & 0 \\ 0 & \sqrt{2}\,i \end{pmatrix}$$

```
plotPhasePlane(A, yBounds, tBounds,M,RHS,eqSoln)
```



II. Solving Systems of First Order Linear Differential Equations in MATLAB

See https://www.mathworks.com/help/symbolic/solve-a-system-of-differential-equations.html for more details

A Stable Arms Race Example

```
syms x(t) y(t)

ode1 = diff(x) == -5*x + 4*y + 10;
ode2 =diff(y) == 3*x - 4*y + 3;
odes = [ode1; ode2 ]
```

odes(t) =

$$\begin{pmatrix} \frac{\partial}{\partial t}\, x(t) = 4\, y(t) - 5\, x(t) + 10 \\ \frac{\partial}{\partial t}\, y(t) = 3\, x(t) - 4\, y(t) + 3 \end{pmatrix}$$

```
S  =dsolve(odes)
```

```
S =
      y: [1x1 sym]
      x: [1x1 sym]
```

```
xSol(t) = S.x
```

xSol(t) =

$$e^{-t}\left(C_{10}+6\,e^{t}\right)-\frac{4\,e^{-8\,t}\left(C_9-\frac{3\,e^{8\,t}}{8}\right)}{3}$$

```
ySol(t) = S.y
```

ySol(t) =

$$e^{-t}\left(C_{10}+6\,e^{t}\right)+e^{-8\,t}\left(C_9-\frac{3\,e^{8\,t}}{8}\right)$$

```
cond1 = x(0) == 0;
cond2 = y(0) == 10;
conds = [cond1; cond2]
```

conds =

$$\begin{pmatrix} x\left(0\right)=0 \\ y\left(0\right)=10 \end{pmatrix}$$

```
[xSol(t), ySol(t)]= dsolve(odes, conds)
```

xSol(t) =

$$e^{-t}\left(6\,e^{t}-\frac{2}{7}\right)+\frac{4\,e^{-8\,t}\left(\frac{3\,e^{8\,t}}{8}-\frac{261}{56}\right)}{3}$$

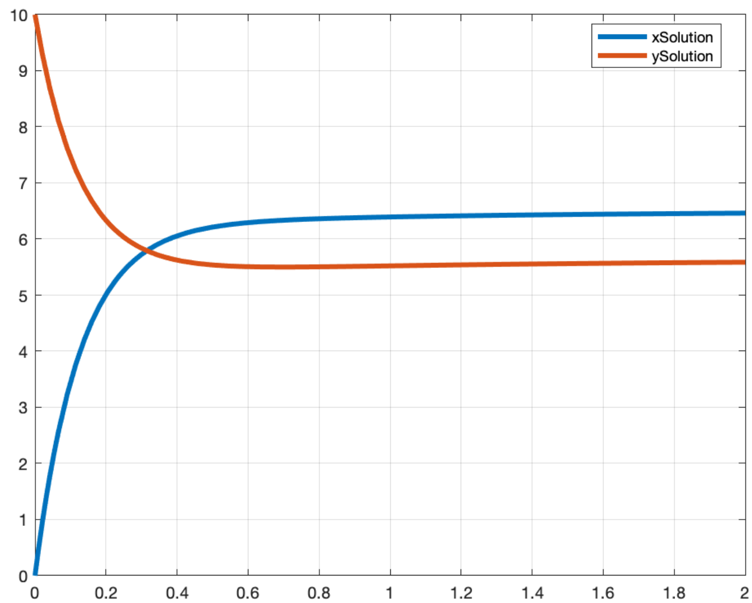
ySol(t) =

$$e^{-t}\left(6\,e^{t}-\frac{2}{7}\right)-e^{-8\,t}\left(\frac{3\,e^{8\,t}}{8}-\frac{261}{56}\right)$$

```
clf
interval = [0,2]
```

```
interval = 1x2
     0     2
```

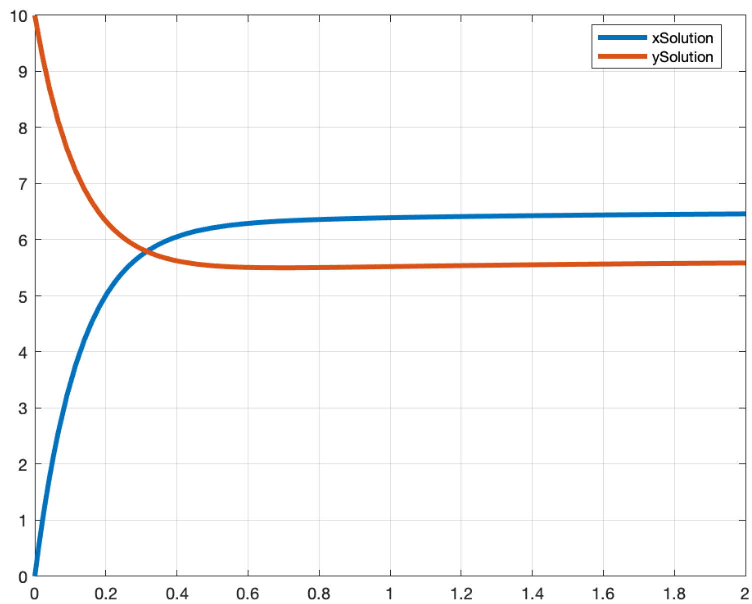
```
fplot(xSol, interval, 'linewidth', 3)
hold on
fplot( ySol, interval, 'linewidth', 3)
grid on
legend( 'xSolution', 'ySolution', 'Location', 'best')
hold on
```




```
```

An Example With Complex Eigenvalues

```
hold off
```




```
ode1 = diff(x) == -.5*x + 1*y;
ode2 =diff(y) == -1*x - .5*y + 3;
odes = [ode1; ode2 ]
```

odes(t) =

$$\begin{pmatrix} \frac{\partial}{\partial t}\, x(t) = y(t) - \frac{x(t)}{2} \\ \frac{\partial}{\partial t}\, y(t) = 3 - \frac{y(t)}{2} - x(t) \end{pmatrix}$$

```
S =dsolve(odes)
```

S =
    y: [1x1 sym]
    x: [1x1 sym]

```
xSol(t) = S.x
```

xSol(t) =

$$e^{-\frac{t}{2}}\cos(t)\left(C_{14} + \frac{12\, e^{t/2}\left(\cos(t) - \frac{\sin(t)}{2}\right)}{5}\right) + e^{-\frac{t}{2}}\sin(t)\left(C_{13} + \frac{12\, e^{t/2}\left(\frac{\cos(t)}{2} + \sin(t)\right)}{5}\right)$$

```
ySol(t) = S.y
```

ySol(t) =

$$e^{-\frac{t}{2}}\,\cos(t)\,\left(C_{13}+\frac{12\,e^{t/2}\,\left(\frac{\cos(t)}{2}+\sin(t)\right)}{5}\right)-e^{-\frac{t}{2}}\,\sin(t)\,\left(C_{14}+\frac{12\,e^{t/2}\,\left(\cos(t)-\frac{\sin(t)}{2}\right)}{5}\right)$$

```
cond1 = x(0) == 0;
cond2 = y(0) == 10;
conds = [cond1; cond2]
```

conds =

$$\begin{pmatrix} x(0)=0 \\ y(0)=10 \end{pmatrix}$$

```
[xSol(t), ySol(t)]= dsolve(odes, conds)
```

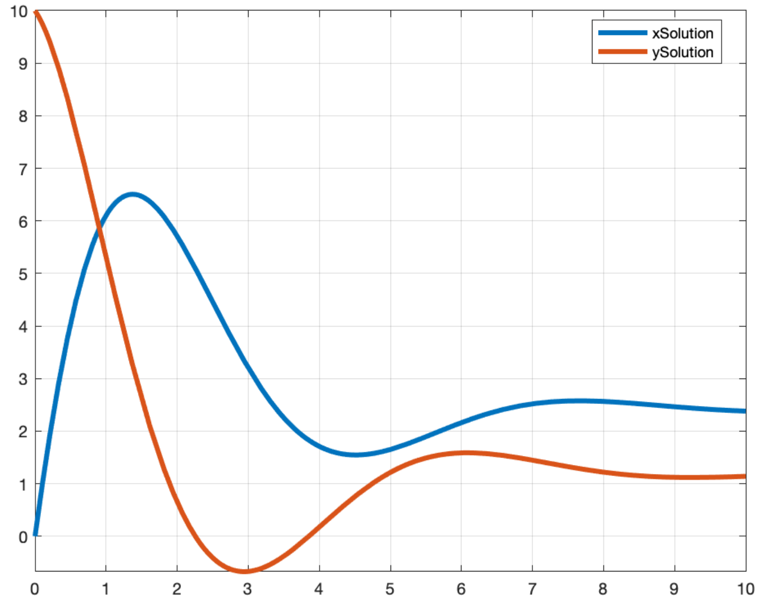
xSol(t) =

$$\frac{44\,\sin(t)}{5\,\sqrt{e^t}}-\frac{12\,\cos(t)}{5\,\sqrt{e^t}}+\frac{12}{5}$$

ySol(t) =

$$\frac{44\,\cos(t)}{5\,\sqrt{e^t}}+\frac{12\,\sin(t)}{5\,\sqrt{e^t}}+\frac{6}{5}$$

```
clf
fplot(xSol, [0 10], 'linewidth', 3)
hold on
fplot( ySol, [0 10], 'linewidth', 3)
grid on
legend( 'xSolution', 'ySolution', 'Location', 'best')
```



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```
A = [1 -1; 1 3 ]
```

```
A = 2x2
    1    -1
    1     3
```

```
RHS = [0;0];
B = rref([A -RHS]);
eqSoln = B(:,end)
```

```
eqSoln = 2x1
    0
    0
```

```
[X D] = eig(sym(A))
```

X =

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

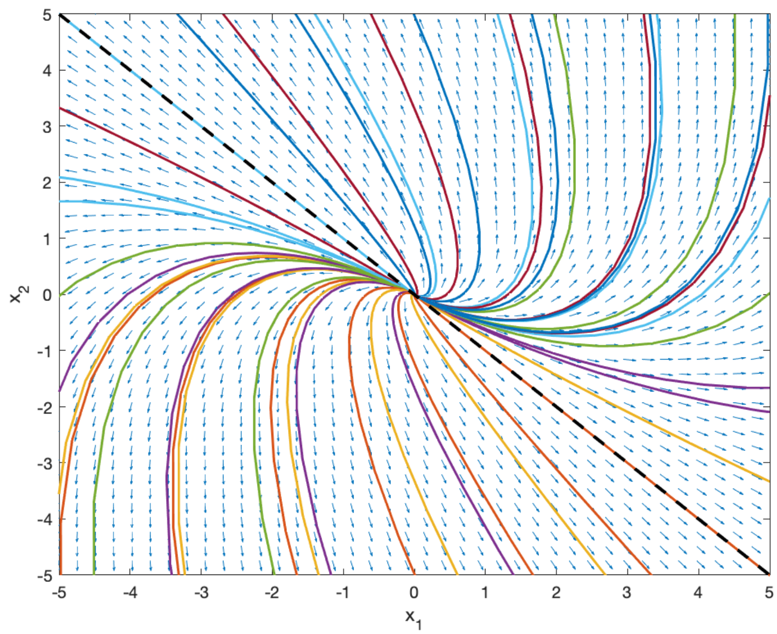
D =

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

```
M = 7;
```

M = 7

```
tBounds = [-50, 50];
plotPhasePlane(A, yBounds, tBounds,M,RHS,eqSoln)
```




```

```

I used a phase plane since no initial conditions were given

```
ode1 = diff(x) == 1*x - 1 *y;
```

```
ode1(t) =
```

$$\frac{\partial}{\partial t}\, x(t) = x(t) - y(t)$$

```
ode2 = diff(y) == 1*x + 3*y;
```

```
ode2(t) =
```

$$\frac{\partial}{\partial t}\, y(t) = x(t) + 3\, y(t)$$

```
odes = [ode1; ode2 ]
```

```
odes(t) =
```

$$\begin{pmatrix} \frac{\partial}{\partial t}\, x(t) = x(t) - y(t) \\ \frac{\partial}{\partial t}\, y(t) = x(t) + 3\, y(t) \end{pmatrix}$$

```


S =dsolve(odes);
```

```
S =
    y: [1x1 sym]
    x: [1x1 sym]
```

```
xSol(t) = S.x
```

```
xSol(t) =
```

$$C_{19}\left(e^{2\,t} - t\, e^{2\,t}\right) - C_{20}\, e^{2\,t}$$

```
ySol(t) = S.y
```

```
ySol(t) =
```

$$C_{20}\, e^{2\,t} + C_{19}\, t\, e^{2\,t}$$

```

```

A System of Three Equations

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```
syms x(t) y(t) z(t)

ode1 = diff(x) == 0*x - 1*y + 2*z;
ode2 =diff(y) == 2*x - 3*y + 2;
ode3 =diff(z) == 3*x -3*y + 1*z
```

```
ode3(t) =
```

$$\frac{\partial}{\partial t} z(t) = 3x(t) - 3y(t) + z(t)$$

```
odes = [ode1; ode2; ode3 ]
```

```
odes(t) =
```

$$\begin{pmatrix} \frac{\partial}{\partial t} x(t) = 2z(t) - y(t) \\ \frac{\partial}{\partial t} y(t) = 2x(t) - 3y(t) + 2 \\ \frac{\partial}{\partial t} z(t) = 3x(t) - 3y(t) + z(t) \end{pmatrix}$$

```
S =dsolve(odes)
```

```
S =
    y: [1x1 sym]
    x: [1x1 sym]
    z: [1x1 sym]
```

```
xSol(t) = S.x
```

```
xSol(t) =
```

$$e^{-t}(C_{23} + e^t) + e^{\sigma_1}\left(\frac{\sqrt{33}}{12} + \frac{1}{4}\right)\left(C_{21} + \frac{2\sqrt{33}\,e^{-\sigma_1}}{11\left(\frac{\sqrt{33}}{2} - \frac{1}{2}\right)}\right) - e^{-\sigma_2}\left(\frac{\sqrt{33}}{12} - \frac{1}{4}\right)\left(C_{22} + \frac{2\sqrt{33}\,e^{\sigma_2}}{11\left(\frac{\sqrt{33}}{2} + \frac{1}{2}\right)}\right)$$

where

$$\sigma_1 = \frac{t\left(\sqrt{33} - 1\right)}{2}$$

$$\sigma_2 = \frac{t\left(\sqrt{33} + 1\right)}{2}$$

```
ySol(t) = S.y
```

```
ySol(t) =
```

$$e^{-t}(C_{23} + e^t) - e^{\sigma_1}\left(\frac{\sqrt{33}}{12} - \frac{3}{4}\right)\left(C_{21} + \frac{2\sqrt{33}\,e^{-\sigma_1}}{11\left(\frac{\sqrt{33}}{2} - \frac{1}{2}\right)}\right) + e^{-\sigma_2}\left(\frac{\sqrt{33}}{12} + \frac{3}{4}\right)\left(C_{22} + \frac{2\sqrt{33}\,e^{\sigma_2}}{11\left(\frac{\sqrt{33}}{2} + \frac{1}{2}\right)}\right)$$

where

$$\sigma_1 = \frac{t\left(\sqrt{33} - 1\right)}{2}$$

$$\sigma_2 = \frac{t\left(\sqrt{33} + 1\right)}{2}$$

```
zSol(t) = S.z
```

zSol(t) =

$$e^{\sigma_1}\left(C_{21} + \frac{2\sqrt{33}\,e^{-\sigma_1}}{11\left(\frac{\sqrt{33}}{2}-\frac{1}{2}\right)}\right) + e^{-\sigma_2}\left(C_{22} + \frac{2\sqrt{33}\,e^{\sigma_2}}{11\left(\frac{\sqrt{33}}{2}+\frac{1}{2}\right)}\right)$$

where

$$\sigma_1 = \frac{t\left(\sqrt{33}-1\right)}{2}$$

$$\sigma_2 = \frac{t\left(\sqrt{33}+1\right)}{2}$$

```
cond1 = x(0) == 1;
cond2 = y(0) == -2;
cond3 = z(0) == 1;
conds = [cond1; cond2; cond3]
```

conds =

$$\begin{pmatrix} x(0) = 1 \\ y(0) = -2 \\ z(0) = 1 \end{pmatrix}$$

```
[xSol(t), ySol(t), zSol(t)] = dsolve(odes, conds)
```

xSol(t) =

$$e^{-t}\left(e^{t}-2\right) - e^{-\sigma_2}\left(\frac{\sqrt{33}}{12}-\frac{1}{4}\right)\left(\frac{2\sqrt{33}\,e^{\sigma_2}}{11\left(\frac{\sqrt{33}}{2}+\frac{1}{2}\right)} - \frac{2\sqrt{33}\left(5\sqrt{33}+3\right)}{33\left(\sqrt{33}+1\right)}\right) + e^{\sigma_1}\left(\frac{\sqrt{33}}{12}+\frac{1}{4}\right)\left(\frac{2\sqrt{33}\,e^{-\sigma_1}}{11\left(\frac{\sqrt{33}}{2}-\frac{1}{2}\right)} + \frac{4\,(2\cdots}{11\,(\sqrt{33}\cdots}\right.$$

where

$$\sigma_1 = \frac{t\left(\sqrt{33}-1\right)}{2}$$

$$\sigma_2 = \frac{t\left(\sqrt{33}+1\right)}{2}$$

ySol(t) =

$$e^{-t}\left(e^{t}-2\right)+e^{-\sigma_2}\left(\frac{\sqrt{33}}{12}+\frac{3}{4}\right)\left(\frac{2\sqrt{33}\,e^{\sigma_2}}{11\left(\frac{\sqrt{33}}{2}+\frac{1}{2}\right)}-\frac{2\sqrt{33}\left(5\sqrt{33}+3\right)}{33\left(\sqrt{33}+1\right)}\right)-e^{\sigma_1}\left(\frac{\sqrt{33}}{12}-\frac{3}{4}\right)\left(\frac{2\sqrt{33}\,e^{-\sigma_1}}{11\left(\frac{\sqrt{33}}{2}-\frac{1}{2}\right)}+\frac{4\left(2\cdots\right.}{11\left(\sqrt{33}\cdots\right.}\right.$$

where

$$\sigma_1=\frac{t\left(\sqrt{33}-1\right)}{2}$$

$$\sigma_2=\frac{t\left(\sqrt{33}+1\right)}{2}$$

```
zSol(t) =
```

$$e^{\sigma_1}\left(\frac{2\sqrt{33}\,e^{-\sigma_1}}{11\left(\frac{\sqrt{33}}{2}-\frac{1}{2}\right)}+\frac{4\left(27\sqrt{33}+11\right)}{11\left(\sqrt{33}-1\right)\left(\sqrt{33}+1\right)}\right)+e^{-\sigma_2}\left(\frac{2\sqrt{33}\,e^{\sigma_2}}{11\left(\frac{\sqrt{33}}{2}+\frac{1}{2}\right)}-\frac{2\sqrt{33}\left(5\sqrt{33}+3\right)}{33\left(\sqrt{33}+1\right)}\right)$$

where

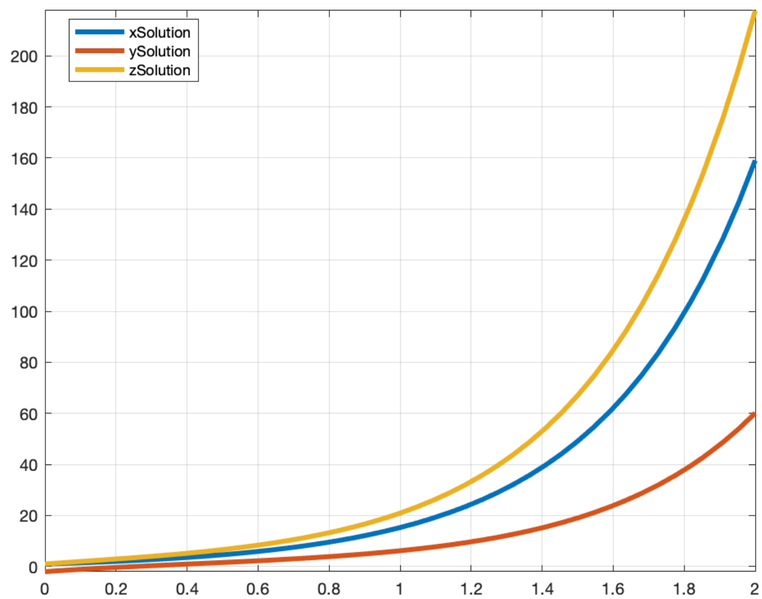
$$\sigma_1=\frac{t\left(\sqrt{33}-1\right)}{2}$$

$$\sigma_2=\frac{t\left(\sqrt{33}+1\right)}{2}$$

```
clf
interval = [0,2]
```

```
interval = 1x2
     0     2
```

```
fplot(xSol, interval, 'linewidth', 3)
hold on
fplot( ySol, interval, 'linewidth', 3)
hold on
fplot( zSol, interval,'linewidth', 3 )
grid on
legend( 'xSolution', 'ySolution', 'zSolution',    ...
    'Location', 'best')
hold on
```


xSolution
ySolution
zSolution
200
180
160
140
120
100
80
60
40
20
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2