MATH 226   Some Notes on Sample Examination 1

1. During much of the 20th Century, commercial whaling fleets killed between 20,000 and 30,000 whales per year. Eventually it became apparent that the numbers being killed was putting the whale population at risk. The International Whaling Commission instituted a ban on commercial whaling in 1986. Only three countries remain who disregard the ban: Iceland, Japan and Norway who annually kill 2,000 whales. Left on their own, whales have a natural growth rate of 2.5 percent.

(a) If W is the number of living whales at time t years, discuss why the differential equation

$$W' = .025W - 2000$$

is a reasonable model for studying the dynamics of the whale population. *The rate of change (W') of the whale population (W) is the net effect of the growth term (.025W) minus the predation term (2000). Derivatives measure rate of change, the growth term is proportional to the population with proportionality constant .025 and the predation term is the number of whales killed per year.*

(b) Find the equilibrium solution of the differential equation. *Equilibrium occurs when* $W' = 0$ *which occurs here when* $.025W - 2000 = 0$; that is, $W = 2000/.025 = 2000/(1/40) = 40\times2000 = 80{,}000$.

(c) Without solving the differential equation, discuss what will happen to whale population in the long term, according to this model, if the present whale population is

(i) 100,000: *W' will initially be positive so whale population will increase and keep increasing without bound.*

(ii) 50,000: *W' will be initially be negative so whale population will decrease and keep decreasing until it goes extinct.*

(d) Sketch a direction field for this differential equation. *See Figure 1 below:*

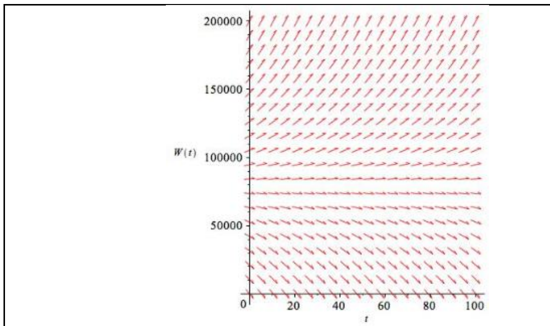


Figure 1

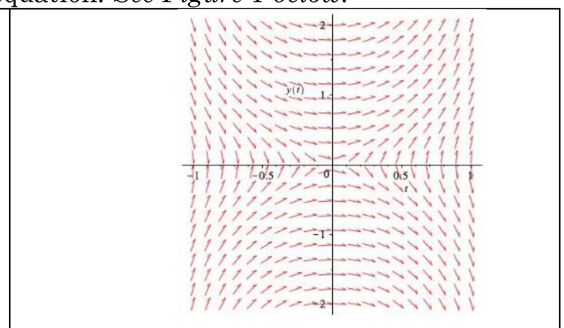


Figure 2

(e) Solve the differential equation by an appropriate change of variable. [Recall that the solution of the differential equation $x' = kx$ is $x = Ce^{kt}$ for some constant C]: *Let x = .025W – 2000 = W/40.- 200 Then x' = W'/40 so W' = 40x' W and the original differential equation is transformed into 40x' = x or x' = (1/40)x which has solution* $x = Ce^{.025t}$*.This yields* $.025W - 2000 = Ce^{.025t}$ *or* $W = 80000 + 40Ce^{.025t}$

(f) Solve the differential equation by the technique of separating variables. *Separating variables and integrating gives* $\int \frac{1}{\frac{1}{40}W-2000}dw = \int 1dt$ so $40\ln\left(\frac{1}{40}W - 2000\right) = t + C$ *or* $\ln\left(\frac{1}{40}W - 2000\right) = .025t + C$. *Exponentiating each side yields* $\frac{1}{40}W - 2000 = Ce^{.025t}$ *so* $W = 80000 + 40Ce^{.025t}$.

(g) Treat the differential equation as a first order linear one and solve using an integrating factor. *Rewrite the equation as W' - .025W = -2000 or* $W' - \frac{1}{40}W = 2000$. *The integrating factor is* $e^{\int\frac{-1}{40}dt} = e^{\frac{-1}{40}t} = e^{-.025t}$. *Multiplying the equation by the integrating factor gives* $e^{\frac{-1}{40}t}W' - \frac{1}{40}e^{\frac{-1}{40}t}W = 2000e^{\frac{-1}{40}t}$. *The left hand side is* $\left(e^{\frac{-1}{40}t}W\right)'$ *so we have* $\left(e^{\frac{-1}{40}t}W\right)' = 2000e^{\frac{-1}{40}t}$ *. Integrating each side with respect to t yields* $e^{\frac{-1}{40}t}W = 80000e^{\frac{-1}{40}t} + C$ *or* $W = 80000 + 40Ce^{.025t}$.

2. Consider the differential equation $y\,y' = t(1+y^2)e^{t^2}$

(a) A student claimed that the direction field for this differential equation looks like this: (See Figure 2 above)

(b) What information can you obtain about the nature of solutions of the differential equation (without solving it!) from this direction field ? *Solutions appear to be symmetric across both horizontal and vertical axes. A solution which begins at a positive value for y for a negative t initially decreases reaching a minimum when t = 0 (where y' = 0) and then increases without bound. A solution which begins at a negative value for y for a negative t initially increases reaching a maximum when t = 0 (where y' = 0) and then decreases without bound.*

(c) Solve the differential equation. *We may rewrite the differential equation in the form* $\frac{y}{1+y^2}\frac{dy}{dt} = te^{t^2}$ *so we see that the variables separate and we have* $\int \frac{y}{1+y^2}dy = \int te^{t^2}dt$ *yielding* $\frac{1}{2}\ln(1+y^2) = \frac{1}{2}e^{t^2} + C$*. Thus* $1+y^2 = Ce^{e^{t^2}}$ *so* $y^2 = Ce^{e^{t^2}} - 1$. There are two possible solutions: $y = \pm\sqrt{Ce^{e^{t^2}} - 1}$, *depending on the initial condition. Note that y' is not defined when y = 0 so no solution can cross the horizontal axis.*

(d) Is the behavior of your solution consistent or inconsistent with the conclusions you derived in part (a)? Give a careful explanation. *Yes, they are consistent. The graph of the negative square root is the mirror image of the graph of the square root. The graph is also symmetric across the vertical axis because y(-t) = y(t) since (-t)² = t².*

3. Existence and Uniqueness of Solutions.

(a) Use the theorem on existence and uniqueness for first order linear differential equations, $y' + p(t)y = g(t)$, to determine (without solving the problem) an interval in which the solution of the problem $(4-t^2)y' + 2ty = 3t^2, \quad y(-3) = 1$ is certain to exist. *Rewrite the equation in the standard form* $y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$ *. We see and p and g are continuous for all t except -2 and 2. For the initial* $t_0 = -3$*, a solution is guaranteed to exist in the interval* $(-\infty < t < -2)$

(b) Determine where in the *ty*-plane are the hypotheses of the theorem about existence and uniqueness (Theorem 2.4.2) for general initial value problem for first order differential equations, $y' = f(t,y), y(t_0) = y_0$, satisfied for the particular problem $y' = \sqrt{9-t^2-y^2}, y(1) = 2$.

Here f and $\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{9-t^2-y^2}}$ *are continuous for* $t^2 + y^2 < 9$*, which is the interior of the region by bounded by the circle of radius 3 centered at the origin in the ty-plane. The Theorem applies to any open rectangle containing (1,2) that lies inside that region.*

4. Imagine a medieval world. In this world a Queen wants to poison a King, who has a wine keg with 500 L of his favorite wine. The Queen gives a conspirator a liquid containing 5 g/L of poison, which must be poured slowly into the keg at a rate of 0.5 L/min. The poisoner must also remove the well-stirred mixture at the same rate, so that the keg is not suspiciously full.

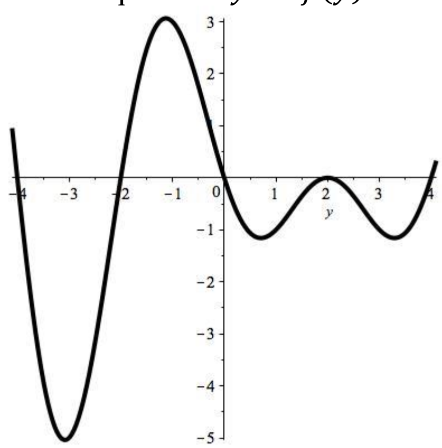
(a) Formulate a differential equation whose solution would be the amount of poison in the keg at each time *t*. Explain your reasoning and carefully label the units on each term.

Let Q(T) be the amount in grams of poison in the keg at time t minutes. Then L(0) = 0 and the rate in is $5\frac{g}{L}\times .5\frac{L}{min} = 2.5\frac{g}{L}$ *while the rate out is* $\frac{Q}{500}\frac{g}{L}\times .5\frac{L}{min} = \frac{Q}{1000}\frac{g}{L}$*. The differential equation is (Rate of change of Q) = (Rate in) – (Rate out) or* $Q' = 2.5 - \frac{Q}{1000}, Q(0) = 0.$

(b) Use your equation from (a) to find the amount of poison in the keg at any time, measured from the start of the pouring by the poisoner. *The differential equation has the same form as the one in Problem 1 and can be solved by any of the 3 techniques to obtain* $Q(t) = 2500(1 - e^{-1000t})$

(c) A plot is hatched for the King to drink wine from the keg while he is on a hunt, where he will become so addled that his prey will surely kill him. The poisoner must pour for a time *T*, when the poison in the keg reaches a dangerous concentration of 0.005 g/L. Find *T*. *When the concentration is 0.005 g/L, the amount of poison is (.005)(500) = 2.5g. We want T so that Q(T) = 2.5 = 5/2. We need to solve* $2500(1 - e^{-1000T}) = 5/2$ *which gives T = 1000 ln* $\left(\frac{1000}{999}\right)$.

5. Population Model (Section 2.5) Identify and characterize all of the equilibrium points of the autonomous first order differential equation $y' = f(y)$where the graph of *f* is shown below:



y	(-∞,-4)	-4	(-4,-2)	-2	(-2,0)	0	(0,2)	2	(2,4)	4	(4,∞)
Sign g(y)	+	0	–	0	+	0	–	0	–	0	+
y moves	→		←		→		←		←		→
		S		U		S		SS		U	

S = asymptotically stable: U = Unstable; SS = semistable

6. Consider the differential equation $t^2y'' + 5ty' + 4y = 0$, $t > 0$

(a) Show that the sum of any two solutions is a solution.
If ϕ and ψ are solutions, then $(\phi + \psi)' = \phi' + \psi'$ and $(\phi + \psi)'' = \phi'' + \psi''$ (by elementary properties of differentiation and $t^2\phi'' + 5t\phi' + 4\phi = 0$ since ϕ is a solution and $t^2\psi'' + 5t\psi' + 4\psi = 0$ since ψ is a solution.
Thus $t^2(\phi + \psi)'' + 5t(\phi + \psi)' + 4(\phi + \psi) = t^2\phi'' + t^2\psi'' + 5t\phi' + 5t\psi' + 4\phi + 4\psi = (t^2\phi'' + 5t\phi' + 4\phi) + (t^2\psi'' + 5t\psi' + 4\psi) = 0 + 0$ so the sum of two solutions is a solution.

(b) Show that any constant multiple of a solution is also a solution. Suppose ϕ is a solution and c is any constant. Then $(c\phi)' = c\,\phi'$ *and* $(c\phi)'' = c\,\phi''$. *Hence* $t^2(c\phi)'' + 5t(c\phi)' + 4(c\phi) = t^2 c\,\phi'' + 5tc\,\phi' + 4c\,\phi = c\,(t^2\phi'' + 5t\,\phi' + 4\,\phi) = c\,0 = 0.$

(c) What does the result of (a) and (b) tell you about the structure of the set of solutions of this differential equation? *The set of solutions is vector subspace of the set of all differential functions.*

(d) Verify that $y_1(t) = t^{-2}$ is a solution of the differential equation.
$y_1' = -2t^{-3}$ *and* $y_1'' = 6t^{-4}$ so $t^2\, y_1'' + 5t\, y_1' + 4\, y_1 = 6t^2t^{-4} - 10tt^{-3} + 4t^{-2}$
$= 6t^{-2} - 10t^{-2} + 4t^{-2} = (6 - 10 + 4)t^{-2} = 0.$

(e) Verify that $y_2(t) = t^{-2}\ln t$ is also a solution of the differential equation.
Here $y_2' = \frac{1}{t^3} - \frac{2lnt}{t^3}$ *and* $y_2'' = \frac{-5}{t^4} + \frac{6\,lnt}{t^4}$ *using the product rule for differentiation. Then so* $t^2\, y_2'' + 5t\, y_2' + 4\, y_2 = \frac{-5}{t^2} + \frac{6\,lnt}{2} + \frac{5}{t^2} - \frac{10lnt}{t^2} + \frac{4lnt}{t^2} = 0$