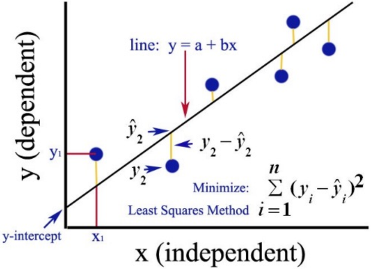
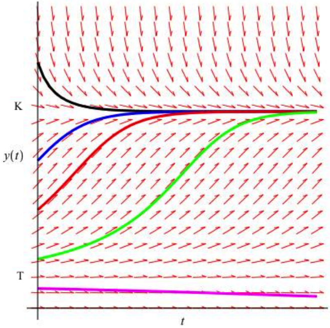
MATH 226: Differential Equations





March 4, 2022

- Notes on Assignment 5
- Revised Assignment 6

Announcements

First Team Project
Due: Next Friday

Announcements

First Team Project
Due: Next Friday

Exam 1: Monday, March 14

Office Hour Today: 12:30 – 1:15

More Announcements

Drop In Tutoring

Sunday: 6 - 7

Monday and Wednesday: 7 - 8

(Room 230, Davis Family Library)

Tutor: Steven Shi

Mathematics Department Lecture

Tuesday at 12:30 (Pizza)

Rose Morris-Wright

Today's Topics

Method of Least Squares

Existence and Uniqueness Theorems for Nonlinear Differential Equations

Qualitative Analysis of Single Species Population Dynamics Autonomous Models

Today's Topics

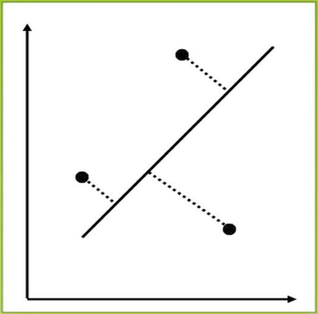
A. Method of Least Squares

B. Existence and Uniqueness Theorems for Linear Differential Equations

The Least Squares Principle

- Regression tries to produce a "best fit equation" --- but what is "best" ?
- Criterion: minimize the **sum of squared deviations** of data points from the regression line.

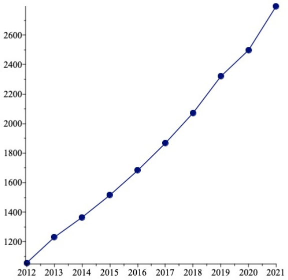
Least Squares



Method of Least Squares

Facebook Users

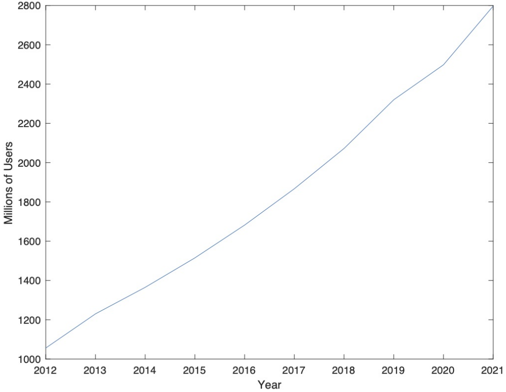
Year	Millions of Users	Year	Millions of Users
2012	1056	2017	1867
2013	1230	2018	2072
2014	1365	2019	2320
2015	1515	2020	2498
2016	1682	2021	2797



Exponential Growth?

MATLAB Code

```
YEARS = [2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021];
USERS = [1056, 1230, 1365, 1515, 1682, 1867, 2072, 2320, 2498, 2797];
plot(YEARS, USERS)
xlabel('Year'), ylabel('Millions of Users')
```



Method of Least Squares

Fitting Data To A Straight Line

Suppose $U(t) = Ce^{at}$

For some constants a and C.

$$\log U(t) = \log C + at$$

Graph of $\log U(t)$ should be a straight line.
How Do We Find a and C?

What Straight Line Best Approximates The Observed Data?

Data: $(x_1, y_1), (x_2, y_2), ...(x_n, y_n)$
Line: $y = ax + b$

Idea: Minimize $\displaystyle\sum_{i=1}^{n}(y_1 - (ax_i + b)))^2$

The Method of Least Squares

The *method of least squares* finds the "best" straight line $y = ax + b$ through a set of data points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ by choosing a and b to minimize

$$\sum_{i=1}^{n}\left(y_i - (ax_i + b)\right)^2,$$

the sum of the squares of the vertical distances between the data and the line. The solution is

$$a = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},\ b = \bar{y} - a\bar{x} \text{ where } \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \text{and } \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Example: Here are six observed data points.

i	1	2	3	4	5	6
x_i	1.2	2.3	3.0	3.8	4.7	5.9
y_i	1.1	2.1	3.1	4.0	4.9	5.9

Use the Method of Least Squares to find the line of best fit. See *Maple* worksheet for assistance

Problem

The table below gives the census data of the United States from 1790 to 1860. Assume that these data fit an exponential growth model $x(t) = x_0 e^{rt}$, or $\log x(t) = \log x_0 + rt$. Use the method of least squares on these data to estimate r. *Hint*: Here $\log x(t)$ corresponds to y, and t corresponds to x.

Note: In *Maple*, you can create a new vector whose entries are the natural logarithms of the entries of another vector by the command

for i **from** 1 to n **do** logx[i] := log(x[i]) **od**

Year	Census Figure
1790	3929214
1800	5236631
1810	7239881
1820	9638453
1830	12866020
1840	17069453
1850	23191876
1860	31443321

Method of Least Squares in *Maple*

Create Two Vectors to Hold the Data.

$x := [1.2, 2.3, 3.0, 3.8, 4.7, 5.9]$

$$[1.2, 2.3, 3.0, 3.8, 4.7, 5.9]$$

$y := [1.1, 2.1, 3.1, 4.0, 4.9, 5.9]$

$$[1.1, 2.1, 3.1, 4.0, 4.9, 5.9]$$

$n := 6$

$$6$$

Use sum Command To Find Average Vales of the Two Variables

$xbar := \left(\frac{1}{n}\right) \cdot sum(x[i], i = 1..n)$

$$3.483333333$$

$ybar := \left(\frac{1}{n}\right) \cdot sum(y[i], i = 1..n)$

$$3.516666667$$

Compute *a* and *r* Using Small Steps

$Denominator := sum\left((x[i] - xbar)^2, i = 1..n\right)$

$$14.26833333$$

$Numerator := sum((x[i] - xbar) \cdot (y[i] - ybar), i = 1..n)$

$$14.99166667$$

$a := \frac{Numerator}{Denominator}$

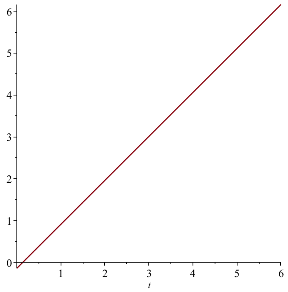
$$1.050695013$$

$b := ybar - a \cdot xbar$

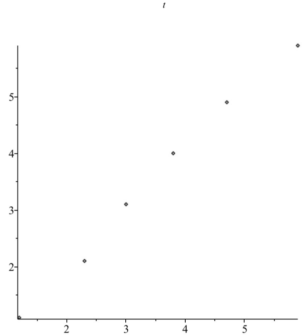
$$-0.143254295$$

Plot Line of Best Fit and Plot the Points (x[i[, y[i])

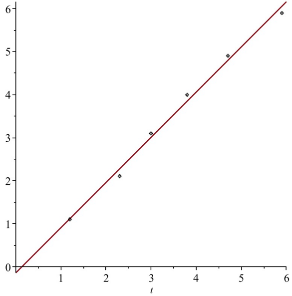
$P1 := plot(a \cdot t + b, t = 0..6)$



with(*plots*) :
P2 := *pointplot*({*seq*([*x*[*i*], *y*[*i*]], i = 1 ..6)})



display(*P1*, *P2*)



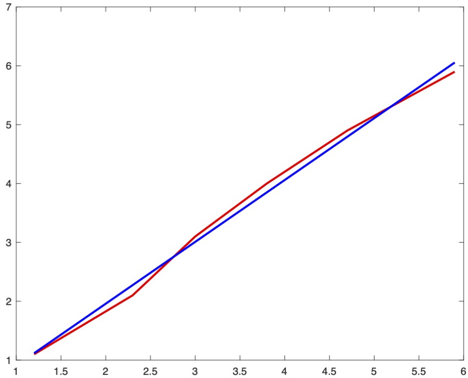
Line of Best Fit in MATLAB

```
x = [1.2, 2.3, 3.0, 3.8, 4.7, 5.9];
y = [1.1, 2.1, 3.1, 4.0, 4.9, 5.9];
n = numel(x) % Number of data points
xsum = sum(x); % add up numbers in x
ysum = sum(y); % % add up numbers in y
xbar = xsum/n; % average x value
ybar = ysum/n; % average y value
A = x - xbar;  % subtract average x value from all x values
B = y - ybar;  % subtract average y value from all y values
Denominator = dot(A,A)% sum of squared entries in A
Numerator = dot(A,B)% sum of the A[i]*B[i]
a = Numer/Denom % slope of line of best fit
b = ybar - a * xbar % intercept of line of best fit
BestFit = a * x +b
plot(x,y, 'r')  % plot of original data
hold on
plot(x,BestFit, 'b') % plot of best fit line
hold off
```

Results:

n = 6 Denominator = 14.2683 Numerator = 14.9917

a = 1.0507 b = -0.1433



7
6
5
4
3
2
1
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6

B.Existence and Uniqueness Theorems

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Notes: Theorem 2.4.1 pertains to **LINEAR** systems

Find longest open interval I containing t_o on which $p(t)$ **and** $g(t)$ are continuous.

You need not find the solution itself to find its interval of definition.

Solutions with different initial conditions do not intersect over their intervals of definition.

B.Existence and Uniqueness Theorems

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PROOF: Follow construction of the solution

$$\phi(t) = \frac{1}{\mu(t)}[\int_{t_o}^{t} \mu(s)g(s)ds + \mu(t_o)y_o]$$

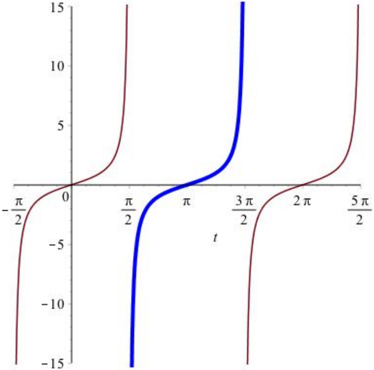
where

$$\mu(t) = e^{\int p(t)dt}$$

Example

$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

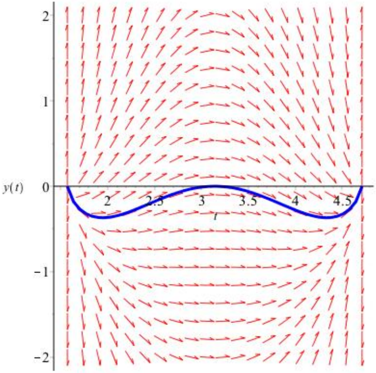
$$g(t) = \sin t$$
is continuous for all t
$$p(t) = \tan t$$
is continuous on $(\pi/2, 3\pi/2)$.



$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

Has Solution

$$y = -\cos t \ln|\cos t|$$



Theorem on Nonlinear Differential Equations

$$y' = f(t, y) \text{ with } y(t_o) = y_o$$

Suppose there is an open rectangle R in (t, y)-plane and (t_o, y_o) is in R.
IF both f and $\partial f/\partial y$ are continuous throughout R,
<u>THEN</u> there is some open interval I centered around t_o and a unique solution $y = \phi(t)$ of the differential equation valid on I with $\phi(t_o) = y_o$

Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

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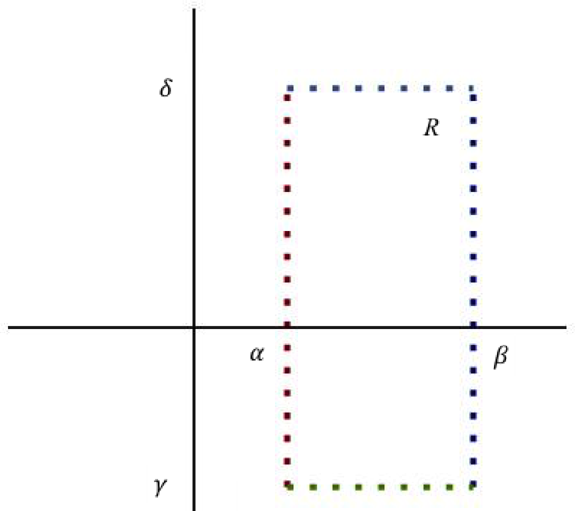

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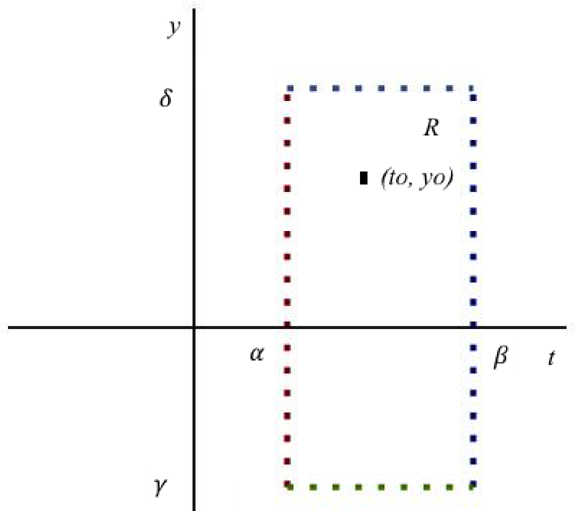

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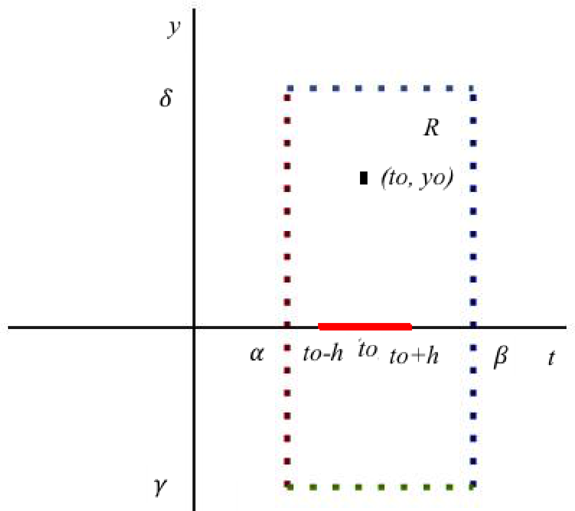


Example

$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

$$f(t,y) = \frac{\ln|ty|}{1 - t^2 + y^2}, \quad \frac{\partial f}{\partial y} = \frac{(1 - t^2 + y^2)(1/y) - \ln|ty|2y}{(1 - t^2 + y^2)^2}$$

Fails to be continuous when
$t = 0$ or $y = 0$ or $1 - t^2 + y^2 = 0$.

Example

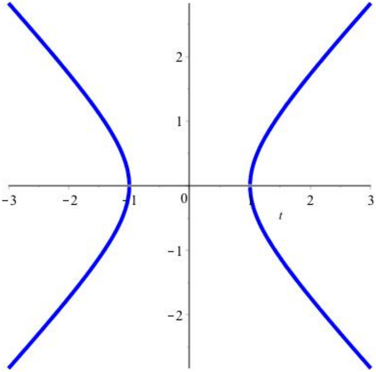
$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

Fails to be continuous when $1 - t^2 + y^2 = 0$ or $y^2 = t^2 - 1$

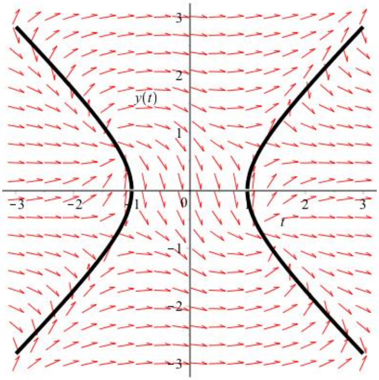
Examine $y^2 = t^2 - 1$.

Must have $t^2 - 1 \geq 0$ so $t^2 \geq 1$

So $t \geq 1$ or $t \leq -1$



$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$



Example

$$y' = y^2 \text{ with } y(0) = 3$$

$f(t,y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous.

Separate Variables

$$y^{-2}y' = 1$$

$$-y^{-1} = t + C$$

Find C: $-3^{-1} = 0 + C$ so $C = -\frac{1}{3}$

$$-\frac{1}{y} = t - \frac{1}{3}$$

$$\frac{1}{y} = -t - \frac{1}{3} = \frac{-3t+1}{3}$$

$$y = \frac{3}{1-3t}$$

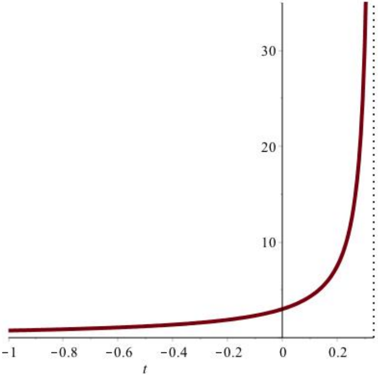
Example

$$y' = y^2 \text{ with } y(0) = 3$$

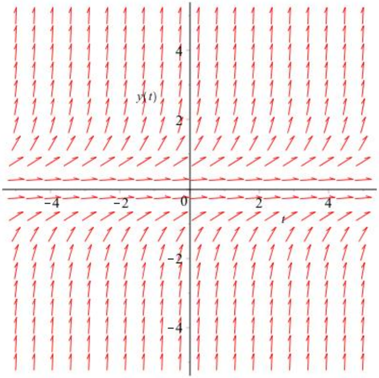
Has Solution

$$y = \frac{3}{1 - 3t}$$

Solution valid on $(-\infty, \frac{1}{3})$



$$y' = y^2$$

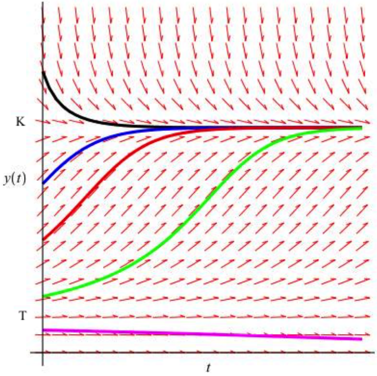


C. Qualitative Analysis of Single Species Population Dynamics Autonomous Models
y' = f(y)

Logistic Growth With Threshold

$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

K is Carrying Capacity and T is Threshold.

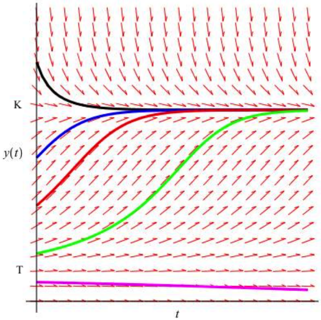


$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

$$y' = 0 \text{ when } y = 0, y = K, y = T$$

$$\begin{array}{ll} y' < 0 : & 0 < y < T \\ y' > 0 & T < y < K \\ y' < 0 : & y > K \end{array}$$

$$y'' = [-\frac{3}{KT}y^2 + (\frac{2}{T} + \frac{2}{K})y - 1]y'$$



Next Time:

Systems of First Order Differential Equations

$$\frac{dx}{dt} = ax(t) + by(t) + f(t)$$

$$\frac{dy}{dt} = cx(t) + dy(t) + g(t)$$

Review Linear Algebra