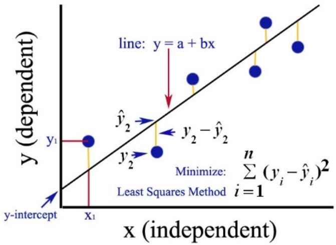
MATH 226: Differential Equations



March 2, 2022

Handouts

- Team Assignments: Project One
- Assignment 6
- Two Fundamental Existence and Uniqueness Theorems
- Method of Least Squares (*Maple* and MATLAB)

Announcements

First Team Project

Due: Week From Friday

Meet With Me

Where to Find the Maple Files

Name	Date Modified	Size	Kind
▶ MATH0223A	Oct 11, 2021 at 3:01 PM	--	Folder
▶ MATH0223B	Oct 11, 2021 at 3:01 PM	--	Folder
▼ MATH0226A	Oct 11, 2021 at 3:01 PM	--	Folder
▶ DROPBOX	Oct 11, 2021 at 3:01 PM	--	Folder
▼ HANDOUTS	Today at 5:55 PM	--	Folder
Assignment4.maple	Today at 3:52 PM	705 KB	Maple...ocument
Class 4 Examples.mw	Feb 20, 2020 at 10:10 AM	556 KB	Maple...ocument
Class 6 Examples.mw	Today at 3:49 PM	557 KB	Maple...ocument
Direction Field for y'=(y-1)(y+1)(y-2).mw	Today at 5:51 PM	93 KB	Maple...ocument
Least Squares in Maple.mw	Today at 5:51 PM	85 KB	Maple...ocument
▶ PUBLIC_HTML	Today at 12:32 AM	--	Folder
▶ RETURN	Feb 9, 2022 at 6:58 AM	--	Folder
▶ SHARE	Oct 11, 2021 at 3:01 PM	--	Folder
▶ WORKSPACE	Feb 9, 2022 at 6:58 AM	--	Folder
▶ MATH0500J	Sep 1, 2021 at 8:38 AM	--	Folder

13. A recent college graduate borrows \$100,000 at an interest rate of 9% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800(1 + t/120)$, where t is the number of months since the loan was made.

(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

Let $S(t)$ be balance due (in dollars) on the loan at time t.
What units should we use for time?
Months or Years?
Note: Interest Rate of 9% is an **annual** rate.
Monthly rate is $\frac{.09}{12}$.
We'll use months as time unit.
Now $S' =$ rate in - rate out
Rate In: What Increases Loan Balance?
Rate In: $\frac{.09}{12}S$
Rate Out: What Decreases Loan Balance?
Rate Out: $800(1 + \frac{t}{120})$

$$S' = \frac{.09}{12}S - 800(1 + \frac{t}{120})$$

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First Order Linear

Solution is $S(t) = \frac{8000}{9}t + \frac{6080000}{27} + Ce^{\frac{3t}{400}}$

With $S(0) = 100000$, $C = -\frac{3380000}{27}$

$$S(t) = \frac{8000}{9}t + \frac{6080000}{27} - \frac{3380000}{27}e^{\frac{3t}{400}}$$

$$S(t) = 888.889t + 225,185.1852 - 125,185.1852e^{\frac{3t}{400}}$$

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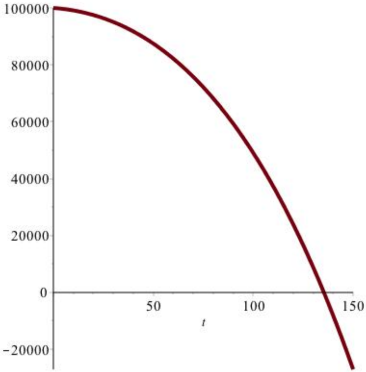

The loan will be paid when $S(t) = 0$ which happens at $t = 135.36$

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly in 20 years?

$$S(t) = 888.889t + 225,185.1852 + Ce^{\frac{3t}{400}}$$

We want to find C so that $S(20 \times 12) = S(240) = 0$
The solution is $C = -72486.62$ so

$$S(t) = 888.889t + 225,185.1852 - 72486.62e^{\frac{3t}{400}}$$

We want $S(0)$ which is 152,698.56.

25. A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5,000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in feet per second.

(a) Find the speed of the skydiver when the parachute opens.
(b) Find the distance fallen before the parachute opens.
(c) What is the limiting velocity v_L after the parachute opens?
(d) Determine how long the skydiver is in the air after the parachute opens.
(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

Distances will be measured in **feet**, time in **seconds**
With $g = 32$ ft/sec/sec, $mg = 180$, so $m = \frac{180}{32} = \frac{45}{8}$ and $\frac{1}{m} = \frac{8}{45}$.

(a) Measure the positive direction downward.
In First 10 seconds, air resistance force is).75v
Newton's Law of Motion: (mass)(acceleration) = sum of forces

$$m\frac{dv}{dt} = -0.75v + mg$$

$$\frac{dv}{dt} = -0.75\frac{v}{m} + g = -\frac{3}{4}\frac{v}{m} + 32 = -\frac{3}{4} \times \frac{8}{45}v + 32$$

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$$\text{Thus } v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Part (a) Continued

$$v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Solution is $v(t) = 240(1 - e^{-\frac{2}{15}t})$

At 10 seconds:

$$v(10) = 240(1 - e^{-\frac{2}{15}(10)}) = 240(1 - e^{-\frac{4}{3}}) \sim 176.7 ft/sec$$

This is the speed of the skydiver when the parachute opens

(b) Find the distance fallen before the parachute opens.
We want $s(10)$ where $s(t)$ is the number of feet fallen after t seconds.
Note: $s(0) = 0$ and $v(t) = s'(t)$ so $s(t) = \int v'(t)dt$
$s(t) = \int 240 - 240e^{-\frac{2}{15}t} = 240t + 1800e^{-\frac{2}{15}t} + C$
Since $s(0) = 0$, we have $C = -1800$.

$$\text{Thus } s(t) = 240t + 1800e^{-\frac{2}{15}t} - 1800$$

Then $s(10) = 600 + 1800e^{-\frac{4}{3}} \sim 1074.47$ feet

(c) What is the limiting velocity v_L **after** the parachute opens?
"Force of air resistance is $12|v|$ when the parachute is open."

$$m\frac{dv}{dt} = -12v + mg \text{ or } v' = -\frac{12}{m}v + 32 = -\frac{32}{15}v + 32$$

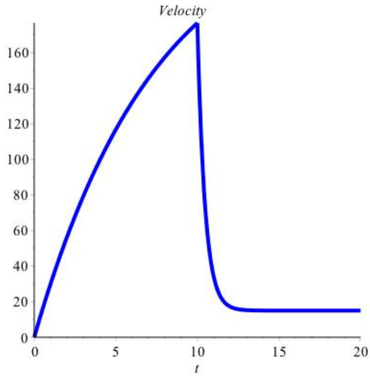
Solution of differential equation is $v(t) = 15 + Ce^{-\frac{32}{15}t}$
Limiting velocity will be 15 ft/sec regardless of the value of C.
If we start the timer again at $t = 0$ when the parachute is open, then $v(0) = 176.7$
This yields a value for C of $C = 161.7$.
Hence $v(t) = 15 + 161.7e^{-\frac{32}{15}t}$

(d) Determine how long the skydiver is in the air after the parachute opens.
Since we know the velocity, we can find the distance $s(t)$ the skydiver has fallen by integrating $v(t)$.

$$s(t) = \int v(t)dt = \int 15 + 161.7e^{-\frac{32}{15}t}dt = 15t - 75.8e^{-\frac{32}{15}t} + D$$

Now use $s(0) = 1074.5$ to find $D = 1150.3$.
The skydiver will hit the ground at time T where $s(T) = 5000$.
The answer is $T = 256.6$ seconds.

(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.



Here are *Maple* commands that would generate this graph:

$f := t \to 240 - 240 * exp(-(2/15) * t)$

$g := t \to 15 + 161.7 * exp(-(32/15) * t)$

$plot(piecewise(t < 10, f(t), g(t-10)), t = 0..20)$

Today's Topics

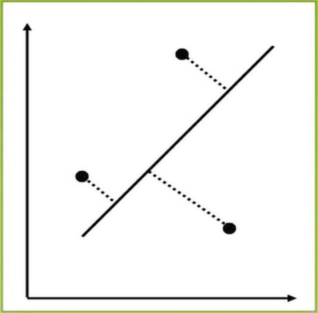
A. Method of Least Squares

B. Existence and Uniqueness Theorems for Linear Differential Equations

The Least Squares Principle

- Regression tries to produce a "best fit equation" --- but what is "best" ?
- Criterion: minimize the **sum of squared deviations** of data points from the regression line.

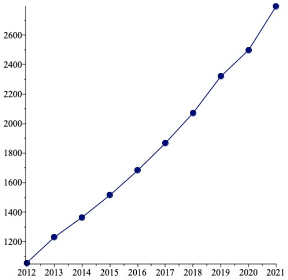
Least Squares



Method of Least Squares

Facebook Users

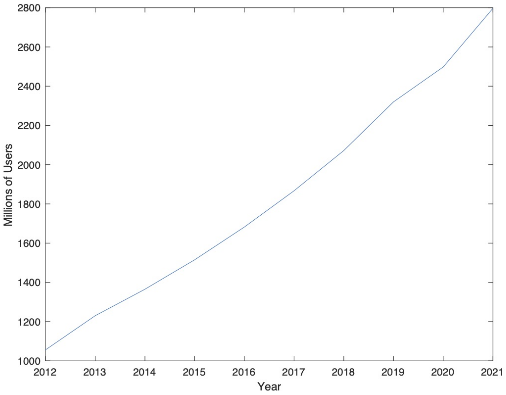
Year	Millions of Users	Year	Millions of Users
2012	1056	2017	1867
2013	1230	2018	2072
2014	1365	2019	2320
2015	1515	2020	2498
2016	1682	2021	2797



Exponential Growth?

MATLAB Code

```
YEARS = [2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021];
USERS = [1056, 1230, 1365, 1515, 1682, 1867, 2072, 2320, 2498, 2797];
plot(YEARS, USERS)
xlabel('Year'), ylabel('Millions of Users')
```



Method of Least Squares

Fitting Data To A Straight Line

Suppose $U(t) = Ce^{at}$

For some constants a and C.

$$\log U(t) = \log C + at$$

Graph of $\log U(t)$ **should be a straight line.**
How Do We Find a **and** C**?**

What Straight Line Best Approximates The Observed Data?

Data: $(x_1, y_1), (x_2, y_2), ...(x_n, y_n)$
Line: $y = ax + b$

Idea: Minimize $\displaystyle\sum_{i=1}^{n}(y_1 - (ax_i + b)))^2$

The Method of Least Squares

The *method of least squares* finds the "best" straight line $y = ax + b$ through a set of data points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ by choosing a and b to minimize

$$\sum_{i=1}^{n}\left(y_i - (ax_i + b)\right)^2,$$

the sum of the squares of the vertical distances between the data and the line. The solution is

$$a = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2},\ b = \overline{y} - a\overline{x} \text{ where } \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \text{and } \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Example: Here are six observed data points.

i	1	2	3	4	5	6
x_i	1.2	2.3	3.0	3.8	4.7	5.9
y_i	1.1	2.1	3.1	4.0	4.9	5.9

Use the Method of Least Squares to find the line of best fit. See *Maple* worksheet for assistance

Problem

The table below gives the census data of the United States from 1790 to 1860. Assume that these data fit an exponential growth model $x(t) = x_0e^{rt}$, or $\log x(t) = \log x_0 + rt$. Use the method of least squares on these data to estimate r. *Hint*: Here $\log x(t)$ corresponds to y, and t corresponds to x.

Note: In *Maple*, you can create a new vector whose entries are the natural logarithms of the entries of another vector by the command

for *i* **from** 1 to *n* **do** logx[i] := log(x[i]) **od**

Year	Census Figure
1790	3929214
1800	5236631
1810	7239881
1820	9638453
1830	12866020
1840	17069453
1850	23191876
1860	31443321

Method of Least Squares in *Maple*

Create Two Vectors to Hold the Data.

$x := [1.2, 2.3, 3.0, 3.8, 4.7, 5.9]$

$[1.2, 2.3, 3.0, 3.8, 4.7, 5.9]$

$y := [1.1, 2.1, 3.1, 4.0, 4.9, 5.9]$

$[1.1, 2.1, 3.1, 4.0, 4.9, 5.9]$

$n := 6$

6

Use sum Command To Find Average Vales of the Two Variables

$xbar := \left(\frac{1}{n}\right) \cdot sum(x[i], i = 1 .. n)$

3.483333333

$ybar := \left(\frac{1}{n}\right) \cdot sum(y[i], i = 1 .. n)$

3.516666667

Compute *a* and *r* Using Small Steps

$Denominator := sum\left((x[i] - xbar)^2, i = 1 .. n\right)$

14.26833333

$Numerator := sum((x[i] - xbar) \cdot (y[i] - ybar), i = 1 .. n)$

14.99166667

$a := \frac{Numerator}{Denominator}$

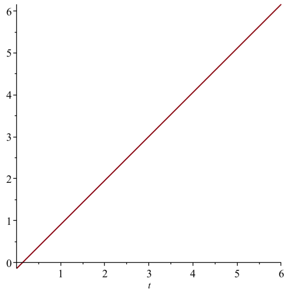
1.050695013

$b := ybar - a \cdot xbar$

-0.143254295

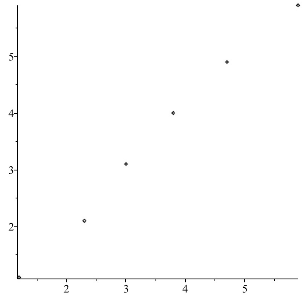
Plot Line of Best Fit and Plot the Points (x[i[, y[i])

$P1 := plot(a \cdot t + b, t = 0 .. 6)$

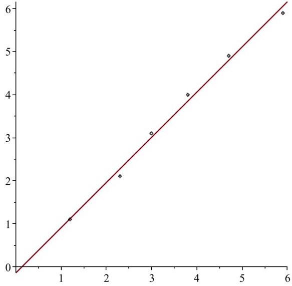


t

with(*plots*) :
P2 := *pointplot*({*seq*([*x*[*i*], *y*[*i*]], i = 1 ..6) })



display(*P1*, *P2*)



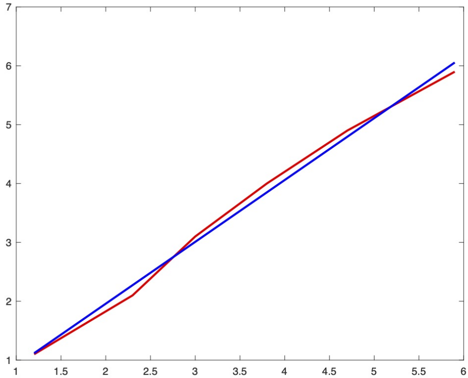
Line of Best Fit in MATLAB

```
x = [1.2, 2.3, 3.0, 3.8, 4.7, 5.9];
y = [1.1, 2.1, 3.1, 4.0, 4.9, 5.9];
n = numel(x) % Number of data points
xsum = sum(x); % add up numbers in x
ysum = sum(y); % % add up numbers in y
xbar = xsum/n; % average x value
ybar = ysum/n; % average y value
A = x - xbar;  % subtract average x value from all x values
B = y - ybar;  % subtract average y value from all y values
Denominator = dot(A,A)% sum of squared entries in A
Numerator = dot(A,B)% sum of the A[i]*B[i]
a = Numer/Denom % slope of line of best fit
b = ybar - a * xbar % intercept of line of best fit
BestFit = a * x +b
plot(x,y, 'r')  % plot of original data
hold on
plot(x,BestFit, 'b') % plot of best fit line
hold off
```

Results:

$n = 6$ Denominator $= 14.2683$ Numerator $= 14.9917$

$a = 1.0507$ $b =$ -0.1433



7
6
5
4
3
2
1
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6