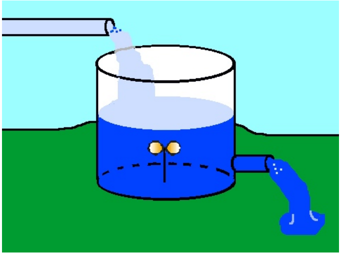
MATH 225: Topics in Linear Algebra and Differential Equations



February 28, 2022

Handouts

Notes on Assignment 4

Assignment 5

Modeling With First Order Differential Equations

Project 1

Project 1 Team Assignments

Announcements

1. First Team Project
   - Today: Team Assignments
   - Week From Thursday: Projects Due
2. Exam 1: Two Weeks From Tonight

Mathematician of the Week I

February 25, 1926 - February 9, 2003

Japanese – Turkish Mathematician

"When you have worked on a problem for a long time without any outcome, leave it alone for a while and consider also some relatively small problems to keep up your courage and productivity. "

MacTutor Biography

Mathematician of the Week II

Lee Alexander Lorch

September 20, 1915 – February 28, 2014)

MacTutor Biography

Mathematicians of the Day

Method of Integrating Factors

Gottfried Leibniz
1646 – 1716
German

Leonhard Euler
1707 – 1783
Swiss

George Stokes
1819 – 1903
English

2. A tank initially contains 200 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 4 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

2. A tank initially contains 200 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 4 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Let $Q(t)$ be the quantity (in grams) of salt in the tank at time t (in minutes)

What are units for $\frac{dQ}{dt}$?

grams/minute

Formulate a Differential Equation Who Solution Will be Q

$$\frac{dQ}{dt} = \text{ rate in} - \text{rate out}$$

Rate IN: Water containing γ grams/liter of salt flows in a rate of 4 liters per minute.

Rate in is:

$$\gamma \frac{grams}{liter} \times 4 \frac{liters}{minute} = 4\gamma \frac{grams}{minute}$$

Let $Q(t)$ be grams of salt in tank at t (in minutes)

$$\frac{dQ}{dt} = \text{ rate in} - \text{rate out}$$

$$\text{Rate in is:} \quad = 4\gamma \frac{grams}{minute}$$

$$\frac{dQ}{dt} = 4\gamma - \text{ rate out}$$

Rate OUT:

$$\frac{Q}{200} \frac{grams}{liter} \times 4 \frac{liters}{minute} = \frac{Q}{50} \frac{grams}{minute}$$

$$\text{Thus } \frac{dQ}{dt} = 4\gamma - \frac{Q}{50}$$

Let $Q(t)$ be grams of salt in tank at t (in minutes)

$$\frac{dQ}{dt} = 4\gamma - \frac{Q}{50}$$

$$\frac{dQ}{dt} + \frac{Q}{50} = 4\gamma$$

The solution is : $Q(t) = 200\gamma + Ce^{-t/50}$
Using $Q(0) = 0$, we get $C = -200\gamma$
Thus $Q(t) = 200\gamma(1 - e^{-t/50})$.
As $t \to \infty, Q(t) \to 200\gamma$.

7. An outdoor swimming pool loses 0.05% of its water volume every day it is in use, due to losses from evaporation and from excited swimmers who splash water. A system is available to continually replace water at a rate of G gallons per day of use.

(a) Find an expression, in terms of G, for the equilibrium volume of the pool. Sketch a few graphs for the volume $V(t)$, including all possible types of solutions.
(b) If the pool volume is initially 1% above its equilibrium value, find an expression for $V(t)$.
(c) What is the replacement rate G required to maintain 12,000 gal of water in the pool?

Let $V(t)$ be the number of gallons of water in the pool at time t days.

Then units on $V' = \frac{dV}{dt}$ are

Gallons/Day

Now $V' =$ rate in - rate out

Rate In:

G.

Rate Out?

0.0005V.

$$\text{So } V' = G - 0.0005V = G - \frac{V}{2000}$$

(a) Equilibrium Volume is:

$V_e = 2000G$

(b) If pool volume is initially 1% above its equilibrium, find an expression for $V(t)$.

The Initial Value Problem is:

$$V' + \frac{V}{2000} = G, V(0) = 1.01V_e = (1.01)(2000)G = 2020G$$

$$V' + \frac{V}{2000} = G, V(0) = 2020G$$

The DE is linear with integrating factor $\mu = e^{t/2000}$.
The general solution is

$$V(t) = 2000G + Ce^{-t/2000}$$

With $V(0) = 2020G$, we get $C = 20G$

and solution is $V = 2000G + 20Ge^{-t/2000}$

What is the replacement rate G required to maintain 12,000 gallons of water in the pool?
$12000 = 2000G$ so $G = 6$ gallons per day.

13. A recent college graduate borrows \$100,000 at an interest rate of 9% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800(1 + t/120)$, where t is the number of months since the loan was made.

(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

Let $S(t)$ be balance due (in dollars) on the loan at time t.

What units should we use for time?

Months or Years?

Note: Interest Rate of 9% is an **annual** rate.

Monthly rate is $\frac{.09}{12}$.

We'll use months as time unit.

Now $S' =$ rate in - rate out

Rate In: What Increases Loan Balance?

Rate In: $\frac{.09}{12}S$

Rate Out: What Decreases Loan Balance?

Rate Out: $800(1 + \frac{t}{120})$

$$S' = \frac{.09}{12}S - 800(1 + \frac{t}{120})$$

$$S' = \frac{.09}{12}S - 800(1 + \frac{t}{120})$$

First Order Linear

Solution is $S(t) = \frac{8000}{9}t + \frac{6080000}{27} + Ce^{\frac{3t}{400}}$

With $S(0) = 100000$, $C = -\frac{3380000}{27}$

$$S(t) = \frac{8000}{9}t + \frac{6080000}{27} - \frac{3380000}{27}e^{\frac{3t}{400}}$$

$$S(t) = 888.889t + 225,185.1852 - 125,185.1852e^{\frac{3t}{400}}$$

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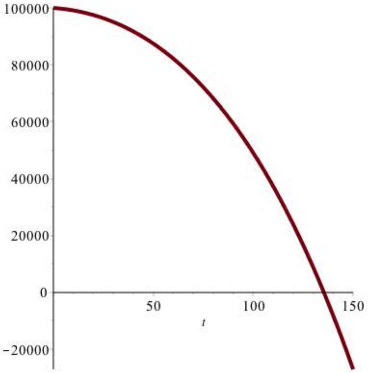


The loan will be paid when $S(t) = 0$ which happens at $t = 135.36$

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly in 20 years?

$$S(t) = 888.889t + 225,185.1852 + Ce^{\frac{3t}{400}}$$

We want to find C so that $S(20 \times 12) = S(240) = 0$
The solution is $C = -72486.62$ so

$$S(t) = 888.889t + 225,185.1852 - 72486.62e^{\frac{3t}{400}}$$

We want $S(0)$ which is 152,698.56.

25. A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5,000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in feet per second.

(a) Find the speed of the skydiver when the parachute opens.
(b) Find the distance fallen before the parachute opens.
(c) What is the limiting velocity v_L after the parachute opens?
(d) Determine how long the skydiver is in the air after the parachute opens.
(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

Distances will be measured in **feet**, time in **seconds**
With $g = 32$ ft/sec/sec, $mg = 180$, so $m = \frac{180}{32} = \frac{45}{8}$
and $\frac{1}{m} = \frac{8}{45}$.

(a) Measure the positive direction downward.
In First 10 seconds, air resistance force is).75v
Newton's Law of Motion: (mass)(acceleration) = sum of forces

$$m\frac{dv}{dt} = -0.75v + mg$$

$$\frac{dv}{dt} = -0.75\frac{v}{m} + g = -\frac{3}{4}\frac{v}{m} + 32 = -\frac{3}{4} \times \frac{8}{45}v + 32$$

$$\frac{dv}{dt} = -\frac{2}{15}v + 32$$

$$\text{Thus } v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Part (a) Continued

$$v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Solution is $v(t) = 240(1 - e^{-\frac{2}{15}t})$

At 10 seconds:

$$v(10) = 240(1 - e^{-\frac{2}{15}(10)}) = 240(1 - e^{-\frac{4}{3}}) \sim 176.7ft/sec$$

This is the speed of the skydiver when the parachute opens

(b) Find the distance fallen before the parachute opens.
We want $s(10)$ where $s(t)$ is the number of feet fallen after t seconds.
Note: $s(0) = 0$ and $v(t) = s'(t)$ so $s(t) = \int v'(t)dt$
$s(t) = \int 240 - 240e^{-\frac{2}{15}t} = 240t + 1800e^{-\frac{2}{15}t} + C$
Since $s(0) = 0$, we have $C = -1800$.

$$\text{Thus } s(t) = 240t + 1800e^{-\frac{2}{15}t} - 1800$$

Then $s(10) = 600 + 1800e^{-\frac{4}{3}} \sim 1074.47$ feet

(c) What is the limiting velocity v_L **after** the parachute opens?
"Force of air resistance is $12|v|$ when the parachute is open."

$$m\frac{dv}{dt} = -12v + mg \text{ or } v' = -\frac{12}{m}v + 32 = -\frac{32}{15}v + 32$$

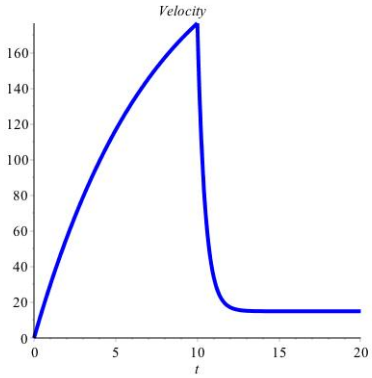
Solution of differential equation is $v(t) = 15 + Ce^{-\frac{32}{15}t}$
Limiting velocity will be 15 ft/sec regardless of the value of C.
If we start the timer again at $t = 0$ when the parachute is open, then $v(0) = 176.7$
This yields a value for C of $C = 161.7$.
Hence $v(t) = 15 + 161.7e^{-\frac{32}{15}t}$

(d) Determine how long the skydiver is in the air after the parachute opens.
Since we know the velocity, we can find the distance $s(t)$ the skydiver has fallen by integrating $v(t)$.

$$s(t) = \int v(t)dt = \int 15 + 161.7e^{-\frac{32}{15}t}dt = 15t - 75.8e^{-\frac{32}{15}t} + D$$

Now use $s(0) = 1074.5$ to find $D = 1150.3$.
The skydiver will hit the ground at time T where $s(T) = 5000$.
The answer is $T = 256.6$ seconds.

(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.



Here are *Maple* commands that would generate this graph:

$f := t \rightarrow 240 - 240 * exp(-(2/15) * t)$

$g := t \rightarrow 15 + 161.7 * exp(-(32/15) * t)$

$plot(piecewise(t < 10, f(t), g(t - 10)), t = 0..20)$

20. Heat transfer from a body to its surroundings by radiation, based on the Stefan–Boltzmann law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4), \qquad \text{(i)}$$

where $u(t)$ is the absolute temperature of the body at time t, T is the absolute temperature of the surroundings, and α is a constant depending on the physical parameters of the body. However, if u is much larger than T, then solutions of Eq. (i) are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4. \qquad \text{(ii)}$$

Suppose that a body with initial temperature 2000 K is surrounded by a medium with temperature 300 K and that $\alpha = 2.0 \times 10^{-12}\ \mathrm{K}^{-3}/\mathrm{s}$.

(a) Determine the temperature of the body at any time by solving Eq. (ii).
(b) Plot the graph of u versus t.
(c) Find the time τ at which $u(\tau) = 600$, that is, twice the ambient temperature. Up to this time, the error in using Eq. (ii) to approximate the solutions of Eq. (i) is no more than 1%.

Next Time:

Differences Between Linear and Nonlinear Differential Equations