MATH 226: Differential Equations

February 23, 2022

Differential Equations So Far

$y' = g(t)$ ($y' =$ expression in t)

$y' = f(y)$ ($y' =$ expression y

$$Today : y' = f(y)g(t)$$

So Far: $y' = g(t), y' = f(y)$

New: **Separable Differential Equation**

$$y' = f(y)g(t)$$

Derivative of y is product of a function of y only and a function of t only.

Example 1: $y' = \dfrac{t^2}{y(1+t^3)} = \dfrac{1}{y}\,\dfrac{t^2}{1+t^3}$

Note: $y \neq 0, t \neq -1$

How To Solve: Separate Variables and Integrate With Respect To Independent Variable

$$y\ y' = \frac{t^2}{(1+t^3)}$$

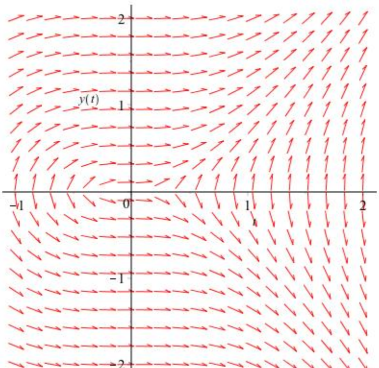
$$y(t)\ y'(t) = \frac{t^2}{(1+t^3)}$$

$$\int y(t)\ y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\int y(t)\, y'(t) = \int \frac{t^2}{(1+t^3)}$$

$$\frac{y^2}{2} = \frac{\ln|1+t^3|}{3} + C$$

$$y^2 = \frac{2}{3}\ln|1+t^3| + C$$



$$\int y(t)\, y'(t) = \int \frac{t^2}{(1+t^3)}$$

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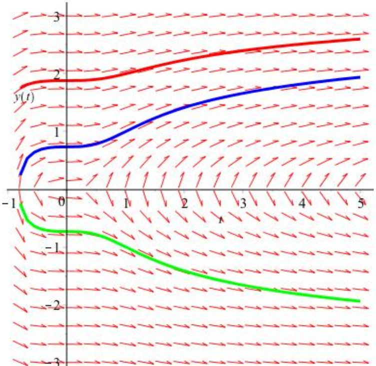


In General

$$y' = f(y)g(t)$$

Is solved as

$$\int \frac{1}{f(y)}\, y' = \int g(t)$$

$$\int \frac{1}{f(y)}\, dy = \int g(t)\, dt$$

Example: An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6$$

$$\int yy' = \int 3 - 2t$$

$$\frac{y^2}{2} = 3t - t^2 + C$$

$$y^2 = 6t - 2t^2 + C$$

$$\text{Set } t = 1, y = -6 :$$

$$36 = 6 - 2 + C \text{ so } C = 32$$

$$y^2 = -2t^2 + 6t + 32$$

Example (Continued): An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6$$

Has Solution

$$y = -\sqrt{-2t^2 + 6t + 32}$$

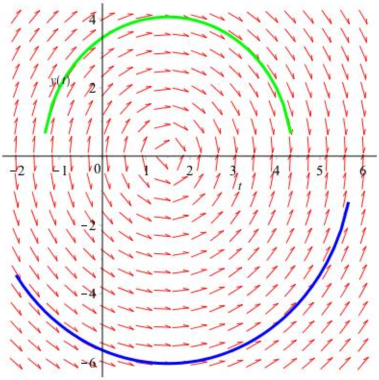
Need $-2t^2 + 6t + 32 = 2(-t^2 + 3t + 16) > 0$
or $t^2 - 3t - 16 < 0$

Roots are $t = \frac{3 \pm \sqrt{9+64}}{2} = \frac{3 \pm \sqrt{73}}{2}$
Solution is valid on $\frac{3-\sqrt{73}}{2} < t < \frac{3+\sqrt{73}}{2}$
Roughly $-2.77 < t < 5.77$.

Example (Continued): An Initial Value Problem

$$y' = \frac{3-2t}{y}, y(1) = -6 \text{ Blue}$$

$$y' = \frac{3-2t}{y}, y(1) = 4 \text{ Green}$$



We can, in principle at least, solve first order differential equations of the form

$$y' = g(t)$$

$$y' = f(y)$$

$$y' = f(y)g(t)$$

We'll Look At One More Type:

$$y'(t) = q(t)y + g(t)$$

First Order LINEAR Differential Equation

Standard Form:

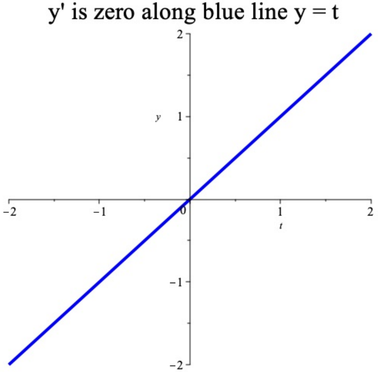
$$y' + p(t)y = g(t)$$

Simple Example: $y' = t - y$

Qualitative Geometric Analysis

Where is $y' = 0$?

$t - y = 0$; that is, the line $y = t$

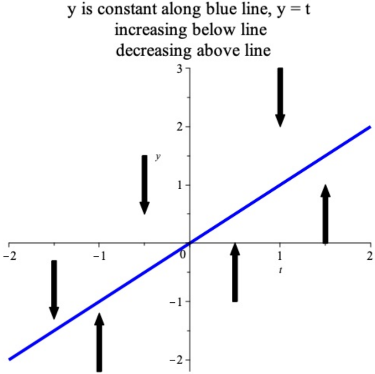


Simple Example: $y' = t - y$

Qualitative Geometric Analysis

Where is $y' > 0$?

$t - y > 0$; that is, $y < t$



Concavity

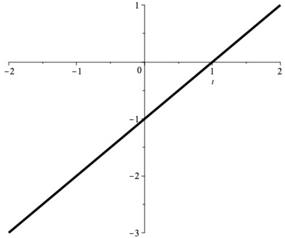
Simple Example: $y' = t - y$

$$y'' = (t-y)' = 1 - y'$$
$$= 1 - (t-y) = 1 - t + y$$

So Graph of Solution y as a function of t is concave up when

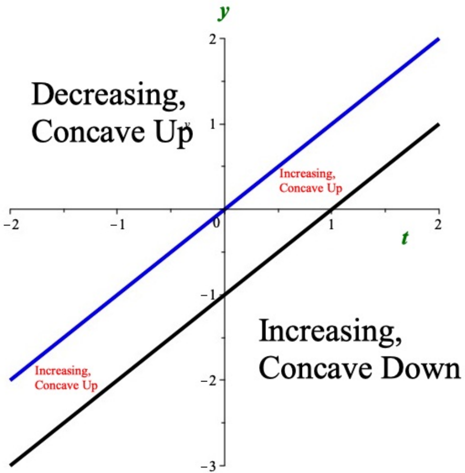
$$1 - t + y > 0$$
$$y > t - 1$$
$$t - y = 0; \text{ that is, the line } y = t$$

Concave Up Above Black Line
Concave Down Below Line



Analysis of: $y' = t - y$

Combine Increasing/Decreasing With Concavity



When analyzing a differential equation, it is useful to look at a plot of the directional fields defined by the equation.

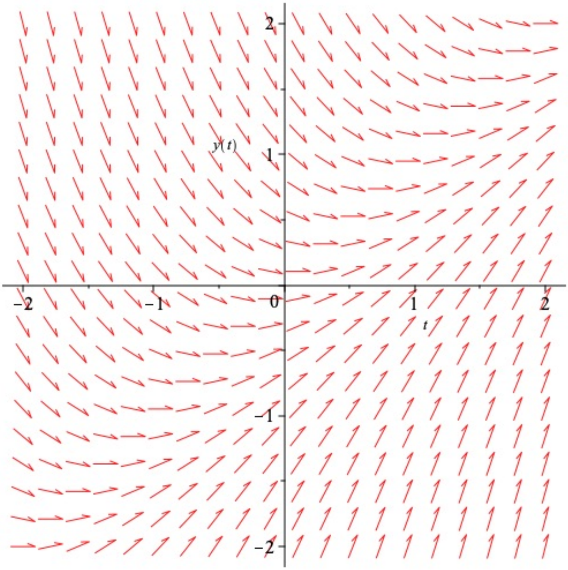
If $y'(t) = f(t, y)$ then at any point (t, y) that a solution passes through, the tangent line to the solution at that point must have slope of $f(t, y)$.

For example, if y'(t) = t -y then a solution passing through the point (2,3) must have slope of 2 -3 = -1 at that point.

To get a nice picture of the general shape of solutions, we can plot a lot of little arrows that have slopes defined by the differential equation.

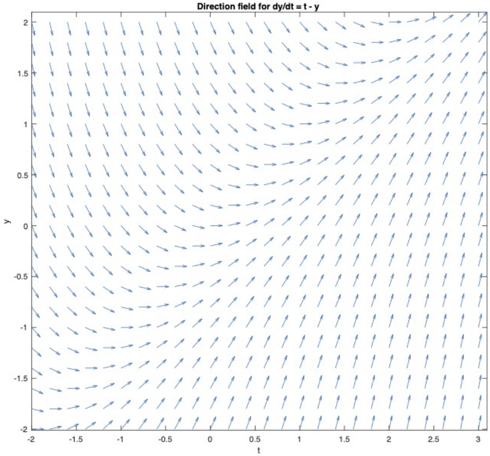
The set of these arrows is called a **direction field** for the differential equation.

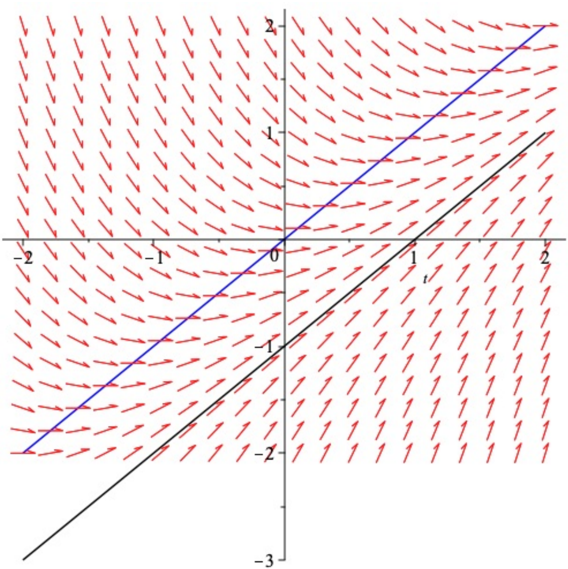
```
with(DEtools) :
ode := y'(t) = t - y(t)
dfieldplot(ode, y(t), t = -2..2, y = -2..2)
```



Plot Direction Field with MATLAB

```
[t, y] = meshgrid(-2:0.2:3, -2:0.2:2);
m = t - y;      % for slope function f = t -
L = sqrt(1 + m.^2);
quiver(t,y, 1./L,s./L, 0.5), axis tight
xlabel 't', ylabel 'y'
title 'Direction field for dy/dt = t - y'
% quiver(t,y, 1./L,s./L, 0.5), axis t
```





How Does $y(t)$ Behave As $t \to \infty$?

Solving $y' = t - y$
We use a Magic Trick

Write as $y' + y = t$

Multiply by e^t : $e^t y' + e^t y = te^t$

Observe $e^t\, y' + e^t y = (e^t\, y)'$

Equation is $(e^t\, y)' = te^t$

Integrate Each Side: $e^t y = (t-1)e^t + C$

Next Time