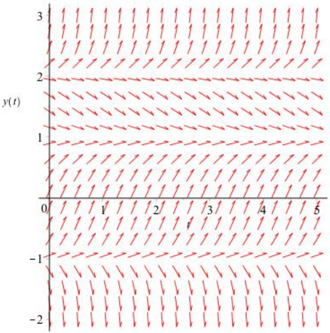
MATH 226: Differential Equations



February 21, 2022

Notes on Assignment 2
Assignment 3
Mumps Model

International Journal of
Environmental Research and Public Health

MDPI

Article

Modelling the Transmission Dynamics and Control of Mumps in Mainland China

Yong Li [1,2,†,‡], Xianning Liu [1,‡,*] and Lianwen Wang [3,‡]

- β: transmission rate,
- λ: loss of vaccination rate,
- ε: vaccine coverage of the susceptible,
- ε_1: vaccine coverage of the exposed,
- κ: invalid vaccination rate,
- α: rate moving from exposed to severe or mild infectious,
- γ: proportion of the severe infections seeking medical advice,
- δ_1: rate moving from severe infectious to recovered,
- δ_2: rate moving from mild infectious to recovered.

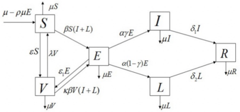


Figure 1. Flowchart of mumps transmission in a population.

$$
\begin{cases}
\frac{dS}{dt} = \mu - \rho\mu E - \beta S(I+L) + \lambda V - (\varepsilon + \mu)S, \\
\frac{dV}{dt} = \varepsilon S + \varepsilon_1 E - \lambda V - \kappa\beta V(I+L) - \mu V, \\
\frac{dE}{dt} = \beta S(I+L) + \rho\mu E + \kappa\beta V(I+L) - (\alpha + \varepsilon_1 + \mu)E, \\
\frac{dI}{dt} = \alpha\gamma E - (\delta_1 + \mu)I, \\
\frac{dL}{dt} = \alpha(1-\gamma)E - (\delta_2 + \mu)L, \\
\frac{dR}{dt} = \delta_1 I + \delta_2 L - \mu R.
\end{cases}
$$

S = Susceptible
V = Vaccinated
E = Exposed
I = Severely Infected
L = Mildly Infected
R = Recovered

Int. J. Environ. Res. Public Health 2018, 15, 33; doi:10.3390/ijerph15010033 www.mdpi.com/journal/ijerph

Disease modelers gaze into their computers to see the future of Covid-19,

Modeling the COVID-19 Epidemic With Multi-Population and Control Strategies

Mathematician of the Week

Margaret H. Wright

Born: February 18, 1944

Silver Professor of Computer Science and former Computer Science Chair, with research interests in optimization, linear algebra, and scientific computing. Member of the National Academy of Science and the National Academy of Engineering. She was awarded the Jon von Neumann prize in 2019 in recognition of her pioneering contributions to the numerical solution of optimization problems and to the exposition of the subject.

Extended Biography

Qualitative Analysis of Autonomous Differential Equation

$$\frac{dy}{dt} = f(y)$$

1. **Find Equilibrium Solutions ($f(y) = 0$)**
2. **Create Phase Line**
   Determine when $f(y) > 0$ and where $f(y) < 0$
   Label with arrows.
3. **Classify Equilibrium Solutions**
   Asymptotically Stable: $\rightarrow \bullet \leftarrow$
   Semistable: $\rightarrow \bullet \rightarrow$ or $\leftarrow \bullet \leftarrow$
   Unstable: $\leftarrow \bullet \rightarrow$

Asymptotically Stable	Semistable	Unstable
$\downarrow$	$\downarrow\uparrow$	$\uparrow$
$\bullet$	$\bullet\ \bullet$	$\bullet$
$\uparrow$	$\downarrow\ \uparrow$	$\downarrow$

4. **Sketch Solutions**
   Increasing, Decreasing, Concavity

Determining Concavity of y as a Function of t

$$y'(t) = f(y(t) \text{ or more simply } y' = f(y)$$

- Use Second Derivative:

$$y''(t) = f'(y(t) \times y'(t) = \frac{df}{dy} \times \frac{dy}{dt} = \frac{df}{dy} \times f(y)$$

- Use Graph of $f(y)$ as a function of y

Determining Concavity of y as a Function of t

Example From Last Time: $y' = (y-1)(y+1)(y-2) = (y^2-1)(y-2)$

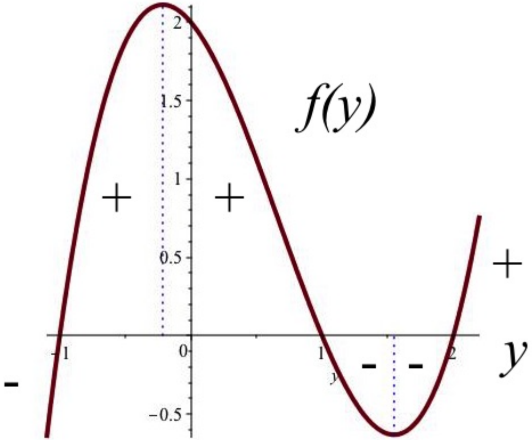
- Use Second Derivative:

$$y''(t) = f'(y(t) \times y'(t) = \frac{df}{dy} \times \frac{dy}{dt} = \frac{df}{dy} \times f(y)$$

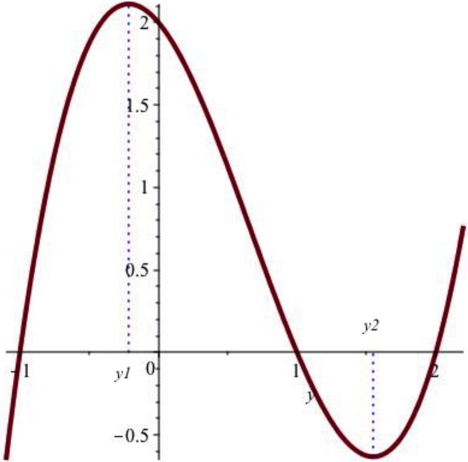
$$y'' = [3y^2 - 4y - 1][(y^2 - 1)(y - 2)]$$

- **Use Graph of $f(y)$ as a function of y**

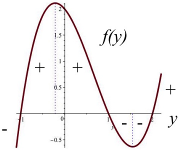
Graph of $f(y) = (y^2 - 1)(y - 2)$ as a function of y



Graph of $f(y) = (y^2 - 1)(y - 2)$ as a function of y



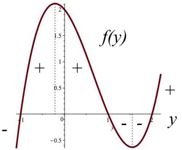
$y1 \sim -.215, y_2 \sim 1.549$



Since $y'(t) = f(y(t))$, we have $y''(t) = \frac{df}{dy}\frac{dy}{dt} = f''(y(t))y'(t)$

Interval of y values	Behavior of Derivative as function of y	$y''(t) = \frac{dy'}{dt} = \frac{dy'}{dy}\frac{dy}{dt}$
$y < -1$	Negative and Increasing	$(+)(\text{-}) = -$
$-1 < y < r_1$	Positive and Increasing	$(+)(+) = +$
$r_1 < y < 1$	Positive and Decreasing	$(\text{-})(+) = -$
$1 < y < y_2$	Negative and Decreasing	$(\text{-})(\text{-}) = +$
$y_2 < y < 2$	Negative and Increasing	$(+)(\text{-}) = -$
$y > 2$	Positive and Increasing	$(+)(+) = +$

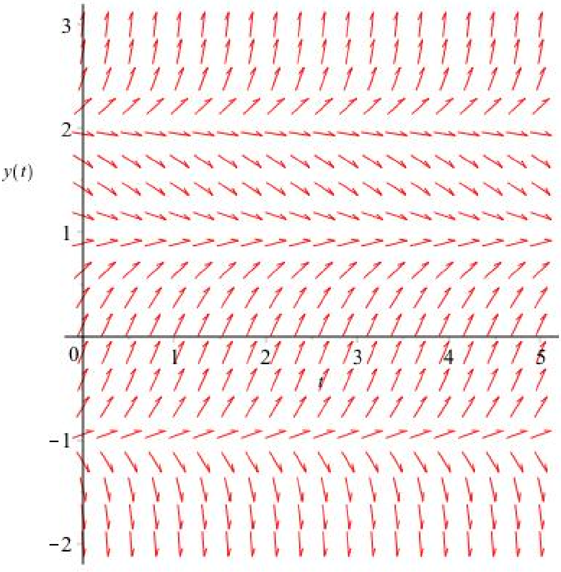
$$y_1 = \frac{2-\sqrt{7}}{3}, y_2 = \frac{2+\sqrt{7}}{3}$$



Since $y'(t) = f(y(t))$, we have $y''(t) = \frac{df}{dy}\frac{dy}{dt} = f''(y(t))y'(t)$

Interval	$y'(t)$	$y''(t)$	Behavior of Solution
$y < -1$	-	-	Decreasing Concave Down
$-1 < y < r_1$	$+$	$+$	Increasing, Concave Up
$r_1 < y < 1$	$+$	$-$	Increasing, Concave Down
$1 < y < y_2$	$-$	$+$	Decreasing, Concave Up
$y_2 < y < 2$	$-$	$-$	Decreasing, Concave Down
$y > 2$	$+$	$+$	Increasing, Concave Up

Direction Field for $y' = (y-1)(y+1)(y-2)$



Direction Field For $y' = (y-1)(y+1)(y-2)$

1) Define the Differential Equation

$DiffEq := y'(t) = (y(t) - 1) \cdot (y(t) + 1) \cdot (y(t) - 2)$

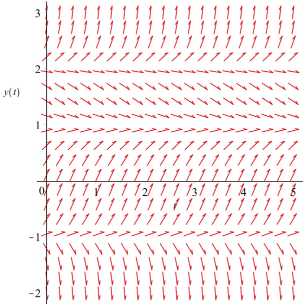
$$D(y)(t) = (y(t) - 1)(y(t) + 1)(y(t) - 2)$$

2) Read in Library of Special Tool

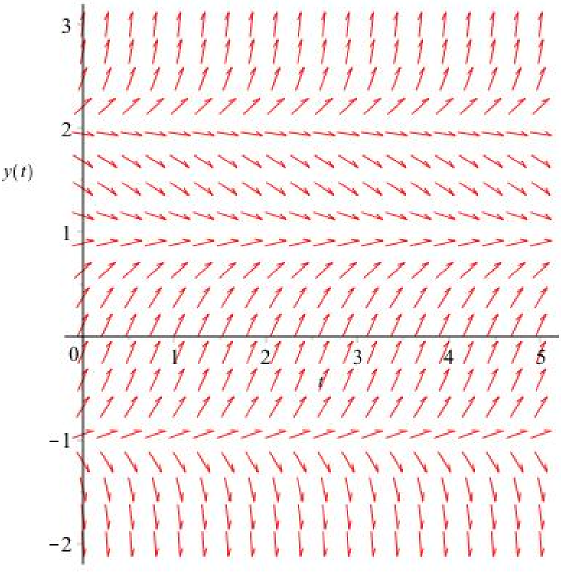
$with(DEtools):$

3) Use *DEplot* Command

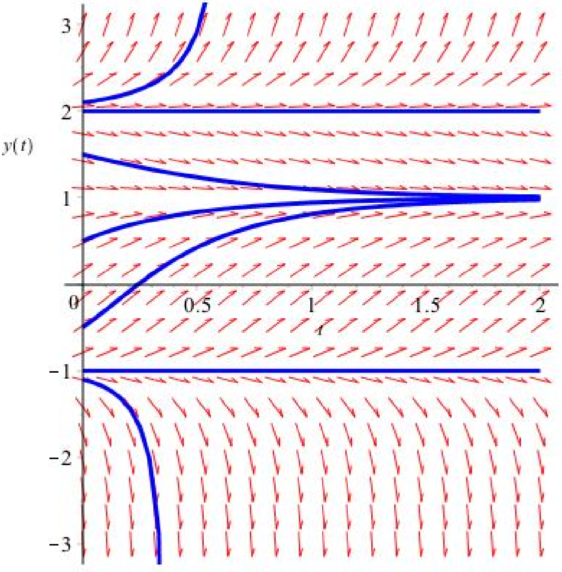
$DEplot(DiffEq, y(t), t = 0..5, y = -2..3)$



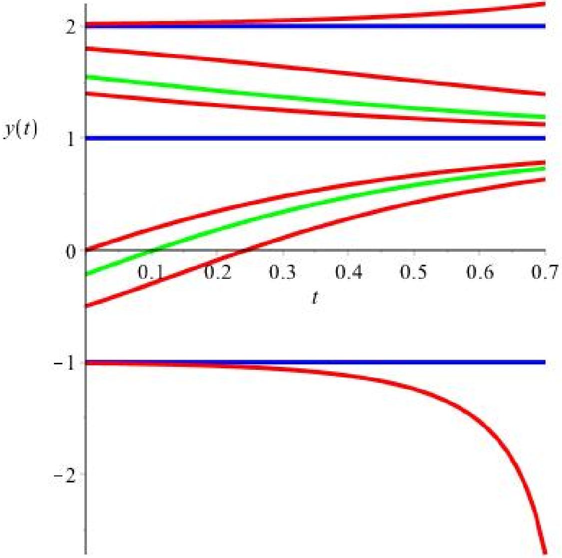
Direction Field for $y' = (y^2 - 1)(y - 2)$



Some Integral Curves For $y' = (y^2 - 1)(y - 2)$



Some More Integral Curves For $y' = (y^2 - 1)(y - 2)$



Another Test For Stability

Theorem: Let y^* be an equilibrium point of $y' = f(y)$ with f having a continuous derivative (as a function of y) in a neighborhood of y^*. Then

- If $f'(y^*) < 0$, then y^* is asymptotically stable
- If $f'(y^*) > 0$, then y^* is unstable
- The test is inconclusive if $f'(y*) = 0$.

So Far: $y' = g(t), y' = f(y)$

New: **Separable Differential Equation**

$$y' = f(y)g(t)$$

Derivative of y is product of a function of y only and a function of t only.