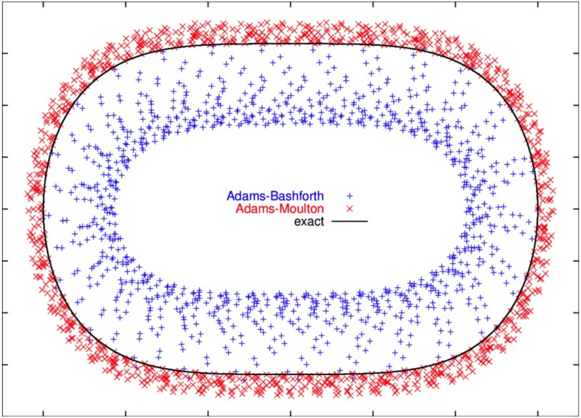
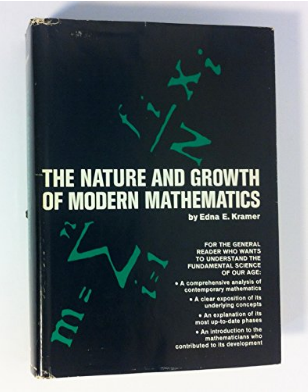
MATH 226 Differential Equations



Class 4: May 11, 2022

Sample Final Exam

Mathematician of the Week



Edna Ernestine Kramer Lassar

May 11, 1902 – July 9, 1984

Announcements

Project 3 Due Friday

Course Response Forms
In Class Next Monday
Bring Laptop/SmartPhone

Final Exam
Friday, May 20: 9 - Noon

Numerical Approximations To Solutions of Differential Equations

$$y' = f(t, y) \text{ with } y(t_0) = y_0$$

Euler's Method: $y_{n+1} = y_n + f(t_n, y_n)$

Improved Euler's:

$$y_{n+1} = y_n + \tfrac{h}{2}\left[f(t_n, y_n)_+ f(t_n + h, y_n + f(t_n, y_n))\right]$$

Classic Runge–Kutta Method (RK4):

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \text{ where}$$

$$k_1 = f(t_n,\ y_n)$$

$$k_2 = f(t_n + \frac{h}{2},\ y_n + \frac{k_1}{2})$$

$$k_3 = f(t_n + \frac{h}{2},\ y_n + h\frac{k_2}{2})$$

$$k_4 = f(t_n + h,\ y_n + hk_3)$$

Start with the Differential Equation

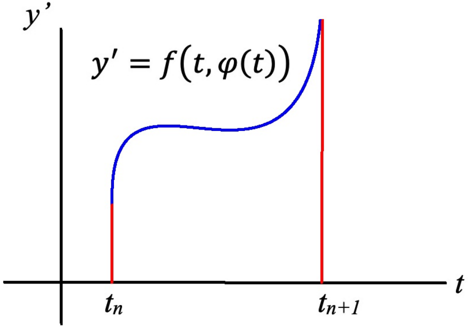
Solution ϕ has $\phi'(t) = f(t_n, \phi(t))$

Integrate both sides over the interval $[t_n, t_{n+1}]$:

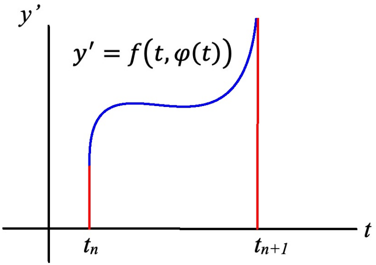
$$\int_{t_n}^{t_{n+1}} \phi'(t)\, dt = \int_{t_n}^{t_{n+1}} f(t_n, \phi(t))\, dt$$

Now left hand side is $\phi(t_{n+1}) - \phi(t_n)$

and right hand side is area under curve



Thus $\phi(t_{n+1}) - \phi(t_n) =$ Area Under Curve



$$\phi(t_{n+1}) = \phi(t_n) + \text{Area Under Curve}$$

$$y_{n+1} = y_n + \text{Area Under Curve}$$

Approximation Schemes Based On Estimating Area Under Curve

$$\int_a^b f(t)\,dt$$

Approach 1: Find Polynomial That Agrees With f At Many Points

- Degree 0 : $P_0(x) = f(a)$
- Degree 1: $P_1(x) : P_1(a) = f(a)$ and $P_1(b) = f(b)$
- Degree 2: $P_2(x) : P_2(a) = f(a), P_2(m) = f(m), P_2(b) = f(b)$ where $m = \frac{a+b}{2}$.

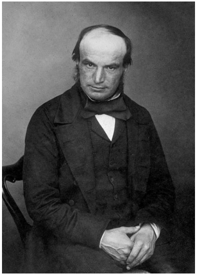
Approximate function by the quadratic polynomial (i.e. parabola) $P(x)$ that takes the same values as the function at the end points a and b and the midpoint $m = (a + b)/2$.

We obtain $P(x) =$

$$f(a)\frac{(x-m)(x-b)}{(a-m)(a-b)} + f(m)\frac{(x-a)(x-b)}{(m-a)(m-b)} + f(b)\frac{(x-m)(x-a)}{(b-a)(b-m)}$$

Lagrange Interpolation Polynomial

$$\text{Then } \int_a^b P(x) = \frac{b-a}{6}\left[f(a) + 4f(m) + f(b)\right]$$



John Couch Adams
June 5, 1819 –
January 21, 1892
Biography

Francis Bashforth
January 8, 1819 –
February 13, 1912
Biography

Forest Ray Moulton
April 29, 1872 –
December 7, 1952
Biography

Cubic Polynomial Approximation

Want $P(a) = f(a), P(p) = f(p), P(q) = f(q), P(b) = b$ where $a < p < q < b$

$$P(x) = f(a)\frac{(x-p)(x-q)(x-b)}{(a-p)(a-q)(a-b)} + f(p)\frac{(x-a)(x-q)(x-b)}{(p-a)(p-q)(p-b)}$$
$$+f(q)\frac{(x-a)(x-p)(x-b)}{(q-a)(q-p)(q-b)} + f(b)\frac{(x-a)(x-p)(x-q)}{b-a)(b-p)(b-q)}$$

Adams-Bashforth Schemes;

$$y_{n+1} = y_n + hf(t_n, y_n)$$

$$y_{n+2} = y_{n+1} + h\left[\frac{3}{2}f(t_{n+1}, y_{n+1}) - \frac{1}{2}f(t_n, y_n)\right]$$

$$y_{n+3} = y_{n+2} + h\left[\frac{23}{12}f(t_{n+2}, y_{n+2}) - \frac{4}{3}f(t_{n+1}, y_{n+1}) + \frac{5}{12}f(t_n, y_n)\right]$$

$$y_{n+4} = y_{n+3} + h\left[\frac{55}{24}f(t_{n+3}, y_{n+3}) - \frac{59}{24}f(t_{n+2}, y_{n+2}) + \frac{37}{24}f(t_{n+1}, y_{n+1}) - \frac{3}{8}f(t_n, y_n)\right]$$

Adams–Moulton Methods are Similar

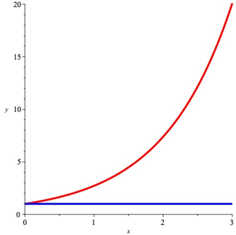
Taylor Series Approach

Choose Polynomial P which agrees with f at a and whose first few derivatives agree with the first few derivatives of f at a:

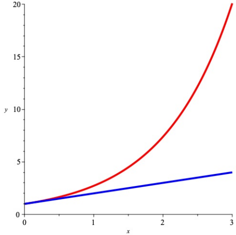
$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \ldots$$

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

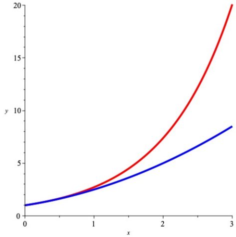
Taylor Series Approximation to e^x on [0,3]:



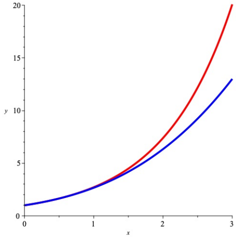
$P(x) = 1$



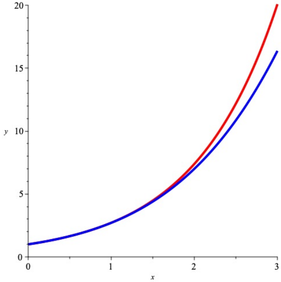
$P(x) = 1 + x$

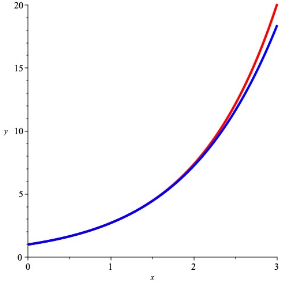


$P(x) = 1 + x + \frac{x}{2}$



$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$





$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

Numerical Methods for Systems of First Order Equations

Write system

$$x' = F(t, x, y), x(t_0) = x_0$$
$$y' = G(t, x, y), y(t_0) = y_0$$

as $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{f} = \begin{pmatrix} F \\ G \end{pmatrix}$$

Euler's Method ($x_{n+1} = x_n + hf_n$) becomes

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n$$

In component form

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h \begin{pmatrix} F(t_n, x_n, y_n) \\ G(t_n, x_n, y_n) \end{pmatrix}$$

Runge-Kutta for Systems

$$\mathbf{x_{n+1}} = \mathbf{x_n} + \frac{h}{6}\left[\mathbf{k_{n1}} = 2\mathbf{k_{n2}} + 2\mathbf{k_{n3}} + \mathbf{k_{n4}}\right]$$

where

$$\mathbf{k_{n1}} = \mathbf{f}\left(t_n, \mathbf{x_n}\right)$$

$$\mathbf{k_{n2}} = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{x_n} + \frac{h}{2}\mathbf{k_{n1}}\right)$$

$$\mathbf{k_{n3}} = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{x_n} + \frac{h}{2}\mathbf{k_{n2}}\right)$$

$$\mathbf{k_{n4}} = \mathbf{f}\left(t_n + h, \mathbf{x_n} + h\mathbf{k_{n3}}\right)$$

Example

$$x' = x + y + t, \text{ with } x(0) = 1$$
$$y' = 4x - 2y, \text{ with } y(0) = 0$$

Euler's Method yields approximate values at
$t_n = 0.2, 0.40.6, 0.8.1.0$ with $h = 0.1$ of

$$\begin{pmatrix} 1.26 \\ 0.76 \end{pmatrix}, \begin{pmatrix} 1.7714 \\ 1.4824 \end{pmatrix}, \begin{pmatrix} 2.58991 \\ 2.3703 \end{pmatrix}, \begin{pmatrix} 3.82374 \\ 3.60413 \end{pmatrix}, \begin{pmatrix} 5.64246 \\ 5.38885 \end{pmatrix}$$

Runge-Kutta yields approximate values at
$t_n = 0.2, 0.40.6, 0.8.1.0$ with $h = 0.1$ of

$$\begin{pmatrix} 1.32489 \\ 0.75916 \end{pmatrix}, \begin{pmatrix} 1.9369 \\ 1.57999 \end{pmatrix}, \begin{pmatrix} 2.93459 \\ 2.66201 \end{pmatrix}, \begin{pmatrix} 4.48422 \\ 4.22784 \end{pmatrix}, \begin{pmatrix} 6.8444 \\ 6.56684 \end{pmatrix}$$

Exact Values are

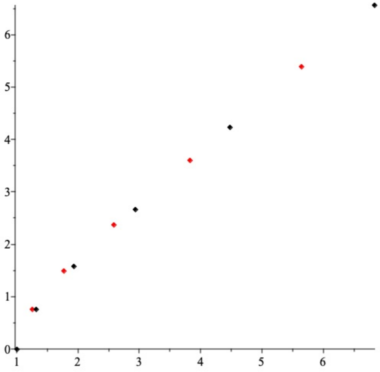
$$\begin{pmatrix} 1.32489 \\ 0.759546 \end{pmatrix}, \begin{pmatrix} 1.93692 \\ 1.58003 \end{pmatrix}, \begin{pmatrix} 2.93463 \\ 2.66208 \end{pmatrix}, \begin{pmatrix} 4.48430 \\ 4.22795 \end{pmatrix}, \begin{pmatrix} 6.84457 \\ 6.84457 \end{pmatrix}$$

$$x' = x + y + t, y' = 4x - 2y \text{ with } x(0) = 1, y(0) = 0$$

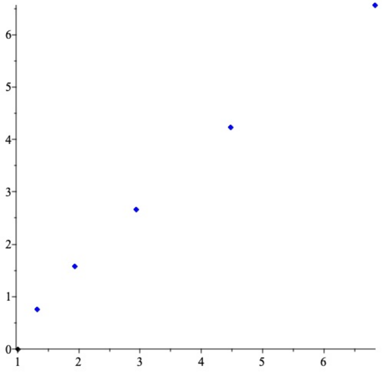
Exact Solution is

$$x(t) = e^{2t} + \frac{2}{9}e^{-3t} - \frac{t}{3} - \frac{2}{9}$$

$$y(t) = e^{2t}. - \frac{8}{9}e^{-3t} = \frac{2t}{3} - \frac{1}{9}$$



Euler vs Exact



Runge-Kutta vs Exact

Next Time

Power Series Solutions

of Differential Equations