MATH 226 Differential Equations

Two Roads Diverged...

Class 30: May 2, 2022

Notes on Assignment 20

Assignment 21

Mathematician of the Week

Samuel Giuseppe Vito Volterra

May 3, 1860 – October 11, 1940

The Fragility of Being a Center

Consider $\mathbf{X'} = A\mathbf{X}$ with $A = \begin{pmatrix} 36 & 80 \\ -50 & -36 \end{pmatrix}$

Characteristic Polynomial: $\lambda^2 + 2704$ so eigenvalues are $\lambda = \pm 52i$

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ϵ small **negative** means real part $\frac{\epsilon}{2} < 0$: Spiral Sink

Bifurcations

First Order System with Parameter

$$x' = F(x, y, \alpha)$$
$$y' = G(x, y, \alpha)$$

where α is a parameter.

How do Critical Points Change as α Varies?

x–nullcline: Where $x' = 0$; that is, $F(x, y, \alpha) = 0$
y–nullcline: Where $x' = 0$; that is, $G(x, y, \alpha) = 0$

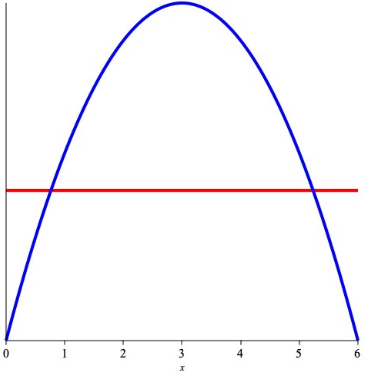
Critical Points: Intersections of x-nullcline and y-nullcline.

Example:

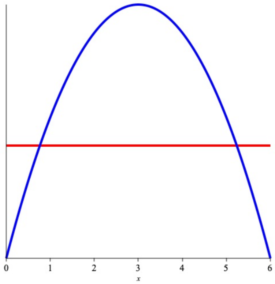
$$x' = F(x, y, \alpha) = -6x + y + x^2$$
$$y' = G(x, y, \alpha) = \frac{3}{2}\alpha - y$$

x-**nullcline:** $y = 6x - x^2 = x(6 - x)$

y-**nullcline:** $y = \frac{3}{2}\alpha$



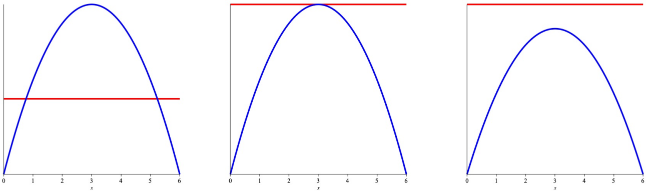
$$x' = -6x + y + x^2, y' \frac{3}{2}\alpha$$



Find Critical Points:

$$6x - x^2 = \frac{3}{2}\alpha$$

$$x = \frac{3 \pm \sqrt{6(6-a)}}{2}$$

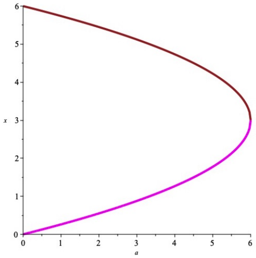


$\alpha = 2$
2 Critical Points

$\alpha = 6$
1 Critical Point

$\alpha = 7$
No Critical Points

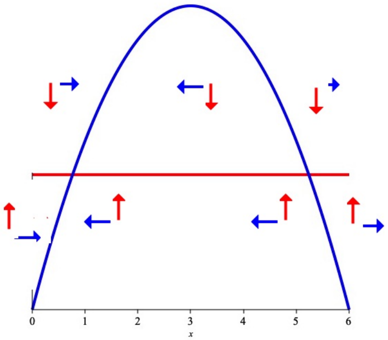
As α varies , the **number** of Critical Points changes.
Graph Location of Critical Points As Functions of α:



$$x = \frac{3 \pm \sqrt{6(6-a)}}{2}$$

As α varies , the **nature** of Critical Points may also change.

$$x' = -6x + y + x^2 \quad y' = \frac{3}{2}\alpha - y$$
$$x' > 0 \qquad y' > 0$$
$$y > 6x - x^2 \qquad y < \frac{3}{2}\alpha$$



Jacobian Matrix

$$x' = F(x,y) = -6x + y + x^2, y' = \frac{3}{2}\alpha - y$$

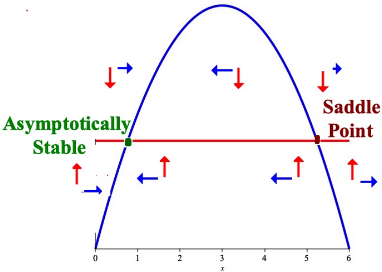
$$J = \begin{pmatrix} F_x & F_y \\ G_x & Gy \end{pmatrix} = \begin{pmatrix} -6+2x & 1 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues: $\{-1, -6+2x\}$

Critical Points: $2x = 3 \pm \sqrt{6(6-a)}$

$2x = 3 - \sqrt{6(6-a)}$: Both Eigenvalues Negative : Asymptotically Stable

$2x = 3 + \sqrt{6(6-a)}$: One Positive, One Negative: Saddle Point



For $\alpha = 6$, there are ONE critical point (3,9)

The Jacobian matrix $\begin{pmatrix} -6+2x & 1 \\ 0 & -1 \end{pmatrix}$ becomes

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

which has eigenvalues of -1 and 0.

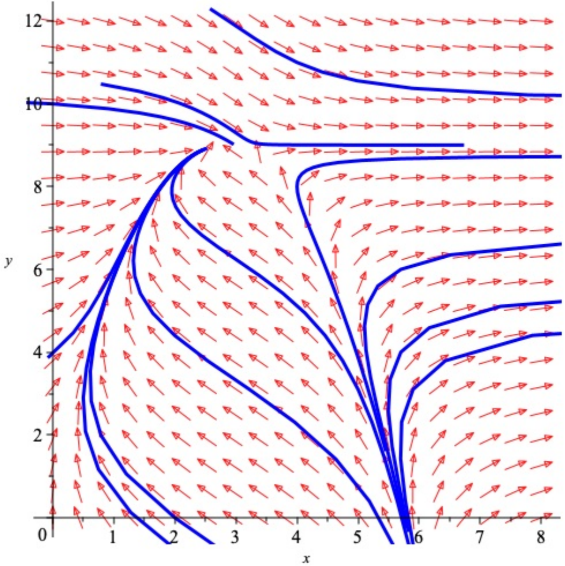
The approximating linear systems is. $x' = y, y' = -y$ which has solutions of the form $x = C_1 + C_2e^{-t}, y = C_2e^{-t}$.

The original system becomes $x' = -6 * x + x^2 + y, y' = 9 - y$.

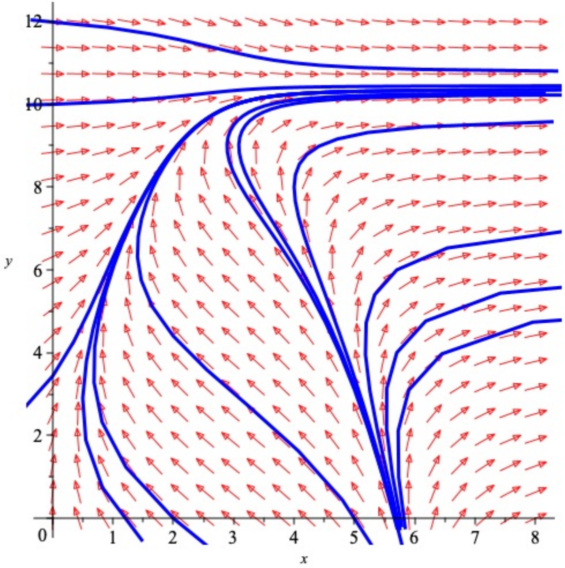
The second equation $y' = 9 - y$ has solution of the form $y = 9 + Ce^{-t}$ so y values will approach 0.

For $\alpha = 6$, there are ONE critical point (3,9)

Here is a Phase Portrait for $\alpha = 6$:



For $\alpha > 6$, there are no critical points.
Here is a Phase Portrait for $\alpha = 7$:



The system $x' = F(x, y, \alpha), y' = G(x, y, alpha)$
As α varies, the configurations of nullclines change. Some possibilities:

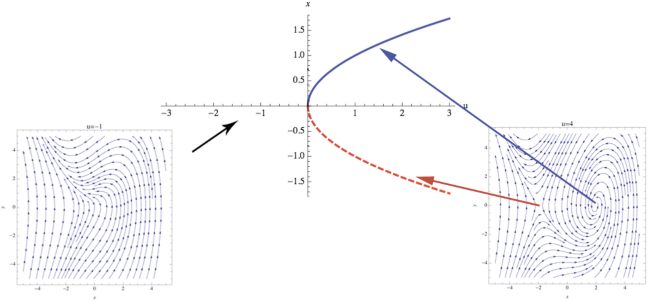
- Two critical points coalesce into one.
- The critical point may disappear
- Two critical points may reappear with different stability properties than before they coalesced.
- We may go from 0 to 1 to 2 critical points

A **bifurcation point** is a value for the parameter α at which critical points coalesce, are gained or are lost.

Bifurcation Diagram

Bifurcation diagrams offer a way to visualize the steady-state stability of differential equations, either individually or in a system, with respect to a given parameter regime and variable.

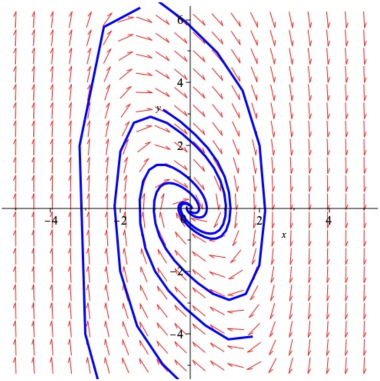
On a standard diagram, the vertical axis will represent the value(s) at which the variable is steady, while the bifurcation parameter will extend horizontally.



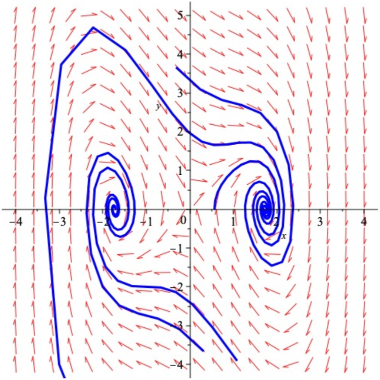
$$x' = -y, y' = x^2 - y - a$$

Pitchfork Bifurcation

$$x' = y, y' = \alpha x - x^3 - y$$

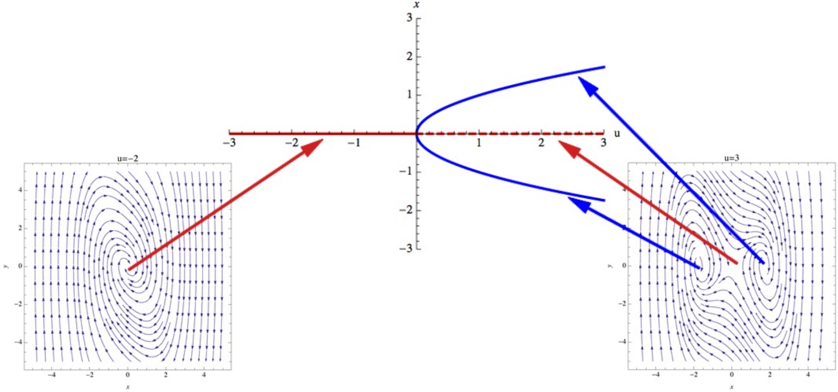


$a = -1$



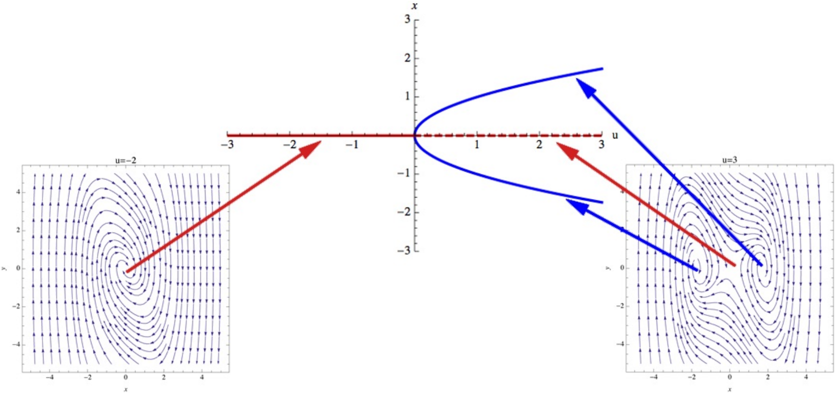
$a = 3$

Pitchfork Bifurcation



Transcritical Bifurcation

While one steady state, (shown in red), remains stationary, the other sweeps across it and, in doing so, trades its stability.



$$x' = y, y' = \alpha x - x^2 - y$$

Transcritical Bifurcation

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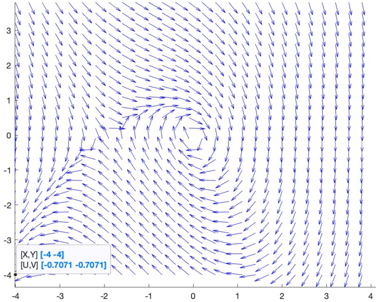
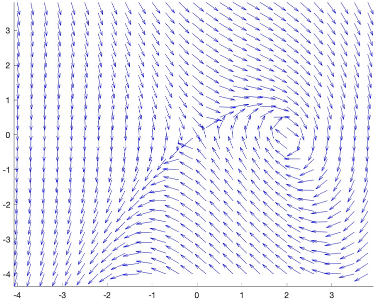
Critical Points: (0,0), $(a, 0)$



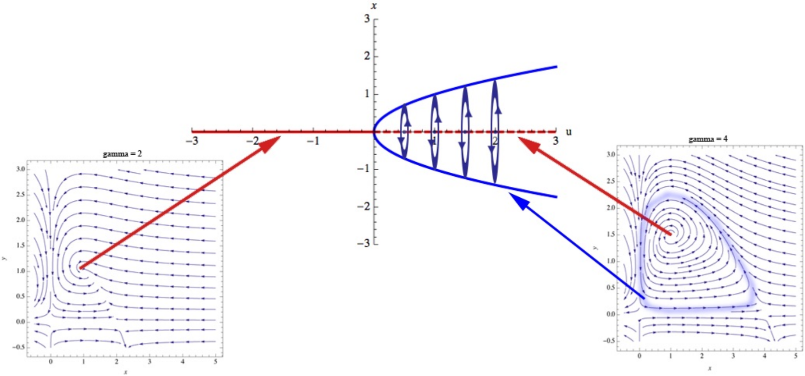
$a = -2$



$a = 2$

Hopf Bifurcation

A Hopf bifurcation occurs when a steady state loses its stability, becoming a source which feeds into a stable periodic orbit.



$$\frac{dx}{dt} = x(1 - \frac{x}{\gamma}) - \frac{xy}{1+x}, \frac{dy}{dt} = \beta(\frac{x}{1+x} - \alpha)y$$

Heinz Hopf

19 November 19, 1894 – June 3, 1971

"Without doubt Heinz Hopf was one of the most distinguished mathematicians of the twentieth century. His work influenced profoundly the evolution not only of topology but of a large part of mathematics. But Heinz Hopf was not only a gifted researcher: he was also an excellent teacher and a personality of the highest integrity; at the same time, he effervesced with charm and subtle humor."

Henri Poincaré. "L'Équilibre d'une masse fluide animée d'un mouvement de rotation". *Acta Mathematica*, volume 7, pages 259-380, September 1885.

SUR L'ÉQUILIBRE D'UNE MASSE FLUIDE
ANIMÉE D'UN MOUVEMENT DE ROTATION

PAR

H. POINCARÉ
À PARIS.

§ 1. *Introduction.*

Quelles sont les figures d'équilibre relatif que peut affecter une masse fluide homogène dont toutes les molécules s'attirent conformément à la loi de NEWTON et qui est animée autour d'un certain axe d'un mouvement de rotation uniforme?

Quelles sont les conditions de stabilité de cet équilibre?

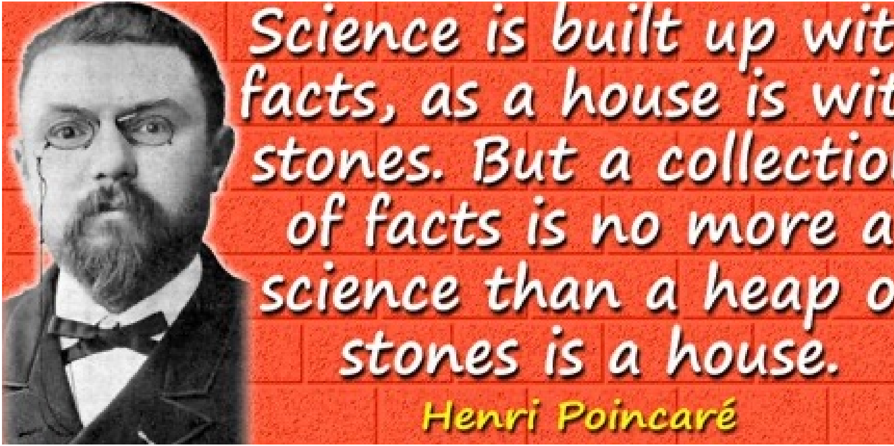
Tels sont les deux problèmes qui forment l'objet de ce mémoire.

On en connaît depuis longtemps deux solutions: l'ellipsoïde de révolution et l'ellipsoïde à trois axes inégaux de JACOBI. Je me propose d'établir qu'il y en a une infinité d'autres.

Mais je vais avant d'aller plus loin signaler un certain nombre de résultats que l'on trouve dans le *Treatise on Natural Philosophy* de MM. TAIT et THOMSON, 2me édition, 778. Sir WILLIAM THOMSON énonce la plupart de ces propositions sans aucune démonstration; pour quelques unes d'entre elles, il renvoie à des mémoires plus étendus insérés aux Philosophical Transactions.

Voici ces résultats, qui doivent nous servir de point de départ.

(*a*). L'ellipsoïde de révolution aplati est une figure d'équilibre toujours stable, si on impose à la masse fluide la condition d'affecter la forme d'un ellipsoïde de révolution.


Science is built up wit
facts, as a house is wit
stones. But a collectio
of facts is no more a
science than a heap o
stones is a house.
Henri Poincaré