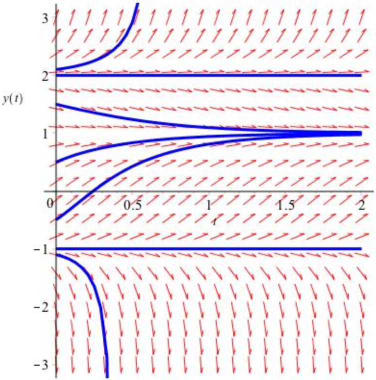
MATH 226: Differential Equations



February 18, 2022

Notes on Assignment 1

Assignment 2
Euler Method Approximation in *Maple* and MATLAB

Announcements

1. No In Person Office Hours Today or Monday
2. Monday's Class on Zoom
3. Getting Started with *Maple*
4. Getting Started with MATLAB

Last Time

$$P' = aP$$

where a is a constant

Qualitative/Graphical Approach

Analytic Approach: $P(t) = P_0 e^{at}$

Numerical Approach: Euler's Method

Exact Solution and Euler Approximation in *Maple*

$ode := P'(t) = .04 \cdot P(t)$

$$ode := \mathrm{D}(P)(t) = 0.04\, P(t) \qquad (1)$$

$init := P(0) = 1000$

$$init := P(0) = 1000 \qquad (2)$$

$dsolve(\ \{ode, init\}\ )$

$$P(t) = 1000\, \mathrm{e}^{\frac{t}{25}} \qquad (3)$$

$A[0] := 1000.0$

$$A_0 := 1000.0 \qquad (4)$$

for i **from** 1 **to** 10 **do** $A[i] := A[i-1] + .04 \cdot A[i-1] \cdot 0.1$ **od**:

for t **from** 0 **to** 10 **do**
  if $t = 0$ **then** *print*('*Time*', '*Approximation*', '*Exact*') **fi** ;
  $print\left(\frac{t}{10.}, A[t], 1000 \cdot \exp\left(\frac{t}{250.0}\right)\right)$
od;

Time, *Approximation*, *Exact*

0., 1000.0, 1000.

0.1000000000, 1004.0000, 1004.008011

0.2000000000, 1008.016000, 1008.032086

0.3000000000, 1012.048064, 1012.072289

0.4000000000, 1016.096256, 1016.128685

0.5000000000, 1020.160641, 1020.201340

0.6000000000, 1024.241284, 1024.290318

0.7000000000, 1028.338249, 1028.395684

0.8000000000, 1032.451602, 1032.517505

0.9000000000, 1036.581408, 1036.655846

1.000000000, 1040.727734, 1040.810774 (5)

A MATLAB Program for Euler Method

```
clc
clear
dt=0.1;
t=0:dt:1.0;
p(1)=1000; % p(0)=1
for i=1:length(t) -1
   p(i+1)=p(i)+dt*.04*p(i);
   fprintf('Approximate value at  t=%f is %f; exact value is %f\n', i/10, p(i+1), 1000* exp(.04 *
i/10) );
end
```

Here is output:

```
Approximate value at  t=0.100000 is 1004.000000; exact value is 1004.008011
Approximate value at  t=0.200000 is 1008.016000; exact value is 1008.032086
Approximate value at  t=0.300000 is 1012.048064; exact value is 1012.072289
Approximate value at  t=0.400000 is 1016.096256; exact value is 1016.128685
Approximate value at  t=0.500000 is 1020.160641; exact value is 1020.201340
Approximate value at  t=0.600000 is 1024.241284; exact value is 1024.290318
Approximate value at  t=0.700000 is 1028.338249; exact value is 1028.395684
Approximate value at  t=0.800000 is 1032.451602; exact value is 1032.517505
Approximate value at  t=0.900000 is 1036.581408; exact value is 1036.655846
Approximate value at  t=1.000000 is 1040.727734; exact value is 1040.810774
```

Today

$$P' = aP + b$$

where a and b are constants

$$P' = aP + b \text{ or } y' = ay + b$$

where a and b are constants

Suppose 5 percent breaks down every month and 1000 units per month flow in.

Let y be amount (in tons) in lake at time t months.

$$y'(t) = -\frac{1}{20}y(t) + 1000 \text{ or } y' = -\frac{1}{20}y + 1000$$

Qualitative Analysis

Begin by looking for Constant Solutions.

What is true about constant functions?

The derivative is always 0.

Find where $y' = 0$.

$$-\frac{1}{20}y + 1000 = 0$$

$$\frac{1}{20}y = 1000$$

$$y = (20)(1000) = 20,000$$

$$y' = 0 \text{ when } y = 20,000$$

Equilibrium, Critical Point, Stationary Point

$$y'(t) = -\frac{1}{20}y(t) + 1000 \text{ or } y' = -\frac{1}{20}y + 1000$$

Qualitative Analysis II

When is y increasing and when is y decreasing?

y increases when y' is positive.

When is $y' > 0$?

Qualitative Analysis of

$$y'(t) = -\frac{1}{20}y(t) + 1000$$

$$\underline{y' > 0}$$

$$-\frac{1}{20}y + 1000 > 0$$

$$1000 > \frac{1}{20}y$$

$$20,000 > y$$

y is increasing when y is less than 20,000.
Similarly: y is decreasing when y is greater than 20,000.
$y' = 0$ when $y = 20,000$ [**Equilibrium, Critical Point, Stationary Point**]

Analytic Solution of $y'(t) = -\frac{1}{20}y(t) + 1000$

Method I: Divide each side by $-\frac{1}{20}y + 1000$ and integrate:

$$\int \frac{y'}{-\frac{1}{20}y + 1000}\, dt = \int 1 dt$$

Method II: Write equation as

$$y' = -\frac{1}{20}[y + \frac{1000}{-\frac{1}{20}}] = -\frac{1}{20}(y - 20,000)$$

and make **Change of Variable** $P = y - 20,000$:

Then $P' = y'$ so the equation becomes

$$P' = -\frac{1}{20}P$$

which has solution $P(t) = P(0)e^{-\frac{1}{20}t}$

$$P' = -\frac{1}{20}P \text{ has solution } P(t) = P(0)e^{-\frac{1}{20}t}$$

where

$$P = y - 20000 \text{ so } y = P + 20000$$

Thus the solution to

$$y'(t) = -\frac{1}{20}y(t) + 1000$$

$$\text{is } y - 20,000 = [y(0) - 20,000]e^{-\frac{1}{20}t}$$

$$\text{or } y = 20,000 + [y(0) - 20,000]e^{-\frac{1}{20}t}$$

$$\text{Since } \lim_{t\to\infty} e^{-\frac{1}{20}t} = 0$$

$$y \to 20,000 \text{ as } t \to \infty$$

Example: If $y(0) = 30,000$, find T so $y(T) = 20,001$.

$$y(t) = 20,000 + [y(0) - 20,000]e^{-\frac{1}{20}t}$$

$$\text{so } 20,001 = 20,000 + [30,000 - 20,000]e^{-\frac{1}{20}T}$$

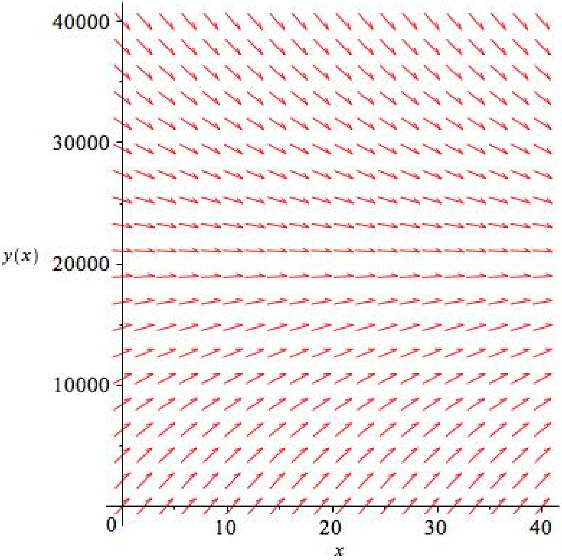
$$1 = 10,000e^{-\frac{1}{20}T}$$

$$e^{-\frac{1}{20}T} = \frac{1}{10,000}$$

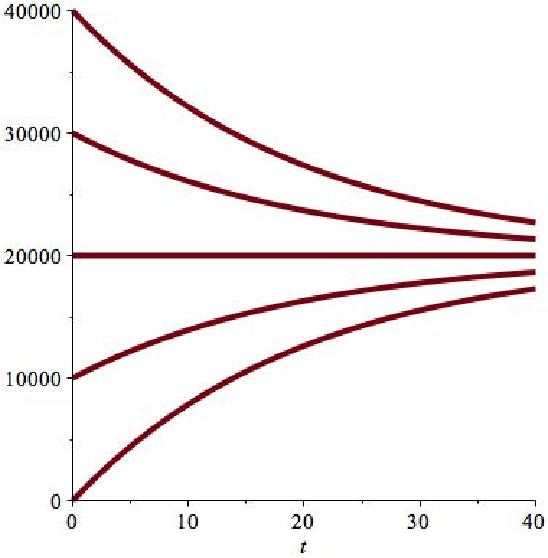
$$-\frac{1}{20}T = \ln(\frac{1}{10,000}) = -\ln 10,000$$

$$T = 20\ln 10,000 = 20\ln 10^4 = 20(4)\ln 10 \sim 184.2$$

Direction Field for $P' = -.05P + 1000$



Some Integral Curves For $P' = -.05P + 1000$



$$\text{Observe } y' = -\frac{1}{20}y + 1000$$

has form $y' = ay + b$ where a, b are constants.
We can solve in the same way:

$$y' = a(y + \frac{b}{a})$$

$$\text{Let } P = y + \frac{a}{b} \text{ so } P' = y'$$

$$\text{We have } P' = aP \text{ so } P = P(0)e^{at}$$

$$\text{or } y + \frac{b}{a} = [y(0) + \frac{b}{a}]e^{at}$$

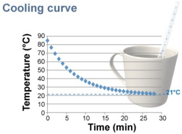
$$y = [y(0) + \frac{b}{a}]e^{at} - \frac{b}{a}$$

WE NOW KNOW HOW TO SOLVE

$$y' = ay + b$$

where a and b are constants,

This equation has been used to model many different situations.



Newton's Law of Cooling

$$y' = -k(y - T) == ky + kt$$

Mice and Owls:

$$P' = 0.5P - 450$$

A Major Generalization

$$y' = ay + b$$

There is no explicit t on right hand side.

More generally, a Differential Equation of the form

$y' = f(y)$ where f is a function only of y

is called an **Autonomous Equation**

Analyzing An Autonomous Differential Equation

Example: $y' = (y^2 - 1)(y - 2) = (y + 1)(y - 1)(y - 2)$

Begin with finding the constant solutions.

They are: $y = -1, y = 1, y = 2$

Next, determine where y is increasing and where it is decreasing by analyzing the sign of y'

Concavity

$$y' = f(y) = (y^2 - 1)(y - 2)$$

$$y'' = f'(y) = (y^2 - 1)(1y') + (2y)y'(y - 2)$$

$$y'' = (y^2 - 1 + 2y^2 - 4y)y'$$

$$y'' = (3y^2 - 4y - 1)y'$$

Concavity

$$y'' = (3y^2 - 4y - 1)y'$$

$$3y^2 - 4y - 1 = 0$$

$$y = \frac{4 \pm \sqrt{16 + 12}}{6} = \frac{4 \pm \sqrt{28}}{6} = \frac{2 \pm \sqrt{7}}{3}$$

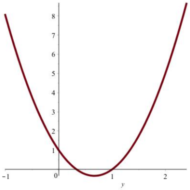
Since $\sqrt{7} \sim 2.5$, roots are about -0.1 and 1.5.

Concavity

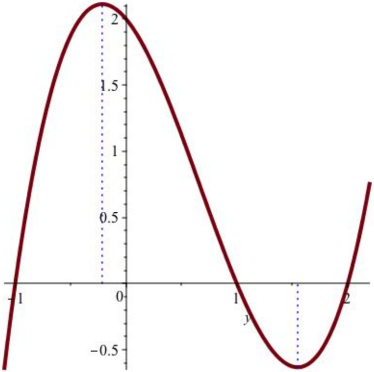
$$y''' = (3y^2 - 4y - 1)y'$$

Note that the graph of $g(y) = 3y^2 - 4y - 1$ is a parabola, opening upward.

y	$g(y) = 3y^2 - 4y - 1$
-1	3 + 4 -1 = 6
+1	3 - 4 -1 = -2
2	12 - 8 - 1 = 3



Graph of $f(y) = (y^2 - 1)(y - 2)$ as a function of y

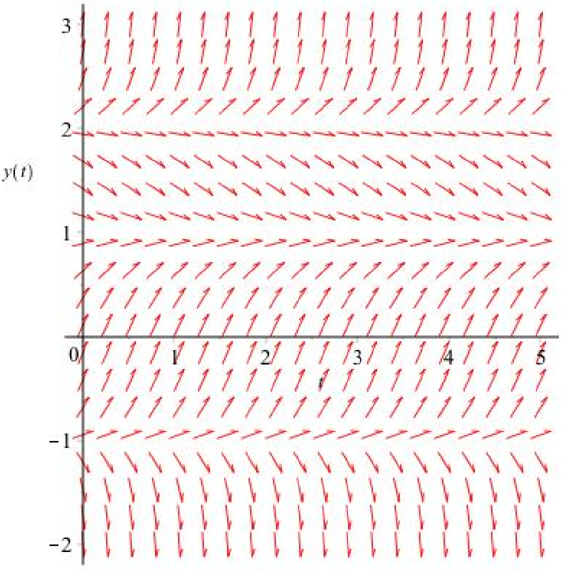


Qualitative Analysis of Autonomous Differential Equation

$$\frac{dy}{dt} = f(y)$$

1. **Find Equilibrium Solutions ($f(y) = 0$)**
2. **Create Phase Line**
   Determine when $f(y) > 0$ and where $f(y) < 0$
   Label with arrows.
3. **Classify Equilibrium Solutions**
   Asymptotically Stable: $\rightarrow \cdot \leftarrow$
   Semistable: $\rightarrow \cdot \rightarrow$ or $\leftarrow \cdot \leftarrow$
   Unstable: $\leftarrow \cdot \rightarrow$
4. **Sketch Solutions**

Direction Field for $y' = (y^2 - 1)(y - 2)$



Some Integral Curves For $y' = (y^2 - 1)(y - 2)$