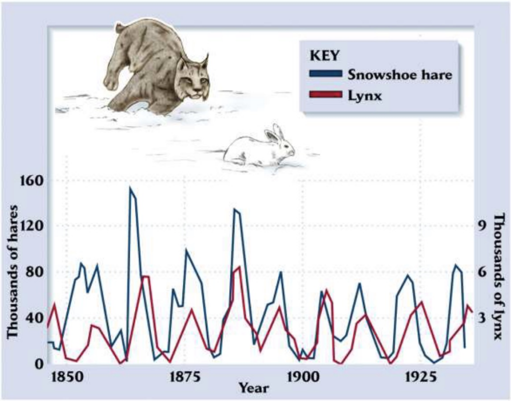
MATH 226: Differential Equations



Class 27: April 25, 2022

Handouts

Notes on Assignment 16
Assignment 17
Optional Extra Credit Problems
Periodic Solutions and Limit Cycles

Mathematician of the Week

R H Bing

October 20, 1914 – April 28, 1986

Schedule This Week

Today	Predator – Prey Model Fragility of a Center
Wednesday	Periodic Solutions and Limit Cycles
Friday	Chaos and Strange Attractors

Current Goal:

Approximating Nonlinear Autonomous System with Linear System Near An Equilibrium Point

$$x' = F(x, y)$$
$$y' = G(x, y)$$

$$F(x, y) \approx F(x^*, y^*) + F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G(x^*, y^*) + G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

But at Equilibrium Points, $F(x^*, y^*) = 0, G(x^*, y^*) = 0$ so

$$F(x, y) \approx F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

$$F(x,y) \approx F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

which we can write as

$$\begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix} \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix}$$

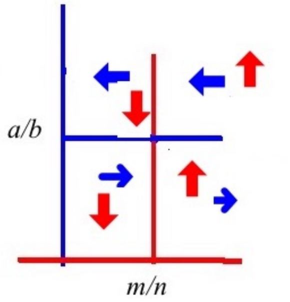
or

$$J(x^*,y^*) \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix} = J(x^*,y^*) \begin{bmatrix} h \\ k \end{bmatrix}$$

J is called the **Jacobi Matrix** or **Jacobian**

Lotka–Volterra Classic Predator – Prey Model

$$x' = ax - bxy = x(a - by)$$
$$y' = -my + nxy = y(-m + nx)$$
$$a, b, m, n > 0$$

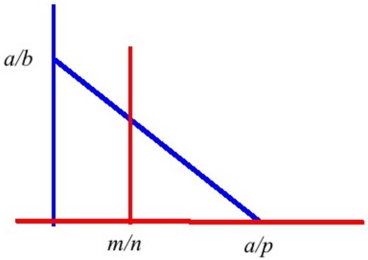


Predator - Prey Model with Logistic Prey Growth

$$x' = ax - px^2 - bxy$$

$$y' = -my + nxy$$

$$a, b, m, n, p > 0$$



$$\frac{a}{p} > \frac{m}{n} \text{ so } n - pm > 0$$

Predator - Prey Model with Logistic Prey Growth

$$x' = ax - px^2 - bxy$$
$$y' = -my + nxy$$
$$a, b, m, n, p > 0$$

$$\begin{aligned} F(x,y) &= ax - px^2 - bxy & G(x,y) &= -my + nx^2 \\ F_x(x,y) &= a - 2px - by & G_x(x,y) &= ny \\ F_y(x,y) &= -bx & G_y(x,y) &= -m + nx \end{aligned}$$

$$J(x,y) = \begin{bmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{bmatrix} = \begin{bmatrix} a - 2px - by & -bx \\ ny & nx - m \end{bmatrix}$$

$$J(x,y) = \begin{bmatrix} a - 2px - by & -bx \\ ny & nx - m \end{bmatrix}$$

At Critical Point $(x^*, y^*) = (\frac{m}{n}, \frac{a}{b} - \frac{pm}{an})$:

$$J(x^*, y*) = \begin{bmatrix} a - 2px - by & -bx \\ ny & nx - m \end{bmatrix}$$

$$A = J(x^*, y^*) = \begin{bmatrix} -\frac{pm}{n} & -\frac{bm}{n} \\ \frac{na-pm}{b} & 0 \end{bmatrix} = \begin{bmatrix} - & - \\ + & 0 \end{bmatrix}$$

so $\text{Trace}(A) < 0$ and $\text{Det}(A) > 0$

$$\lambda = \frac{Trace(A) \pm \sqrt{(Trace(A))^2 - 4Det(A)}}{2}$$

Real Parts of Eigenvalues Are Negative

For Classic Lotka–Volterra Model, set $p = 0$

$$x' = ax - bxy$$
$$y' = -my + nxy$$
$$a, b, m, n, p > 0$$

$$\begin{array}{cc} F(x,y) = ax - bxy & G(x,y) = -my + nx^2 \\ F_x(x,y) = a - by & G_x(x,y) = ny \\ F_y(x,y) = -bx & G_y(x,y) = -m + nx \end{array}$$

$$J(x,y) = \begin{bmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{bmatrix} = \begin{bmatrix} a - by & -bx \\ ny & nx - m \end{bmatrix}$$

$$J(x,y) = \begin{bmatrix} a - by & -bx \\ ny & nx - m \end{bmatrix}$$

At Critical Point $(x^*, y^*) = (\frac{m}{n}, \frac{a}{b})$:

$$J(x^*,y^*) = \begin{bmatrix} a - by & -bx \\ ny & nx - m \end{bmatrix}$$

$$A = J(x^*,y^*) = \begin{bmatrix} 0 & -\frac{bm}{n} \\ \frac{na}{b} & 0 \end{bmatrix} = \begin{bmatrix} 0 & - \\ + & 0 \end{bmatrix}$$

so Trace$(A) = 0$ and Det$(A) > 0$

$$\lambda = \frac{Trace(A) \pm \sqrt{(Trace(A))^2 - 4Det(A)}}{2} = \pm \frac{\sqrt{-4Det(A)}}{2}$$

so $\lambda = \pm i\sqrt{det(A)}$. Eigenvalues are Pure Imaginary.

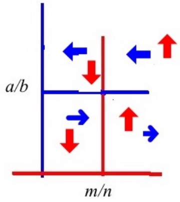
Jacobian Analysis of Classic Lotka – Volterra

$$F(x,y) = ax - bxy \qquad F_x = a - by \qquad F_y = -bx$$
$$G(x,y) = -my + nxy \qquad G_x = ny \qquad G_y = -m + nx$$

$$J(x,y) = \begin{pmatrix} a - by & -bx \\ ny & -m + nx \end{pmatrix}$$
$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -m \end{pmatrix}$$

Eigenvalue $\lambda = a$ with eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Eigenvalue $\lambda = -m$ with eigenvector $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Jacobian Analysis of Classic Lotka – Volterra

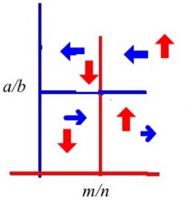
$$J(x,y) = \begin{pmatrix} a - by & -bx \\ ny & -m + nx \end{pmatrix}$$

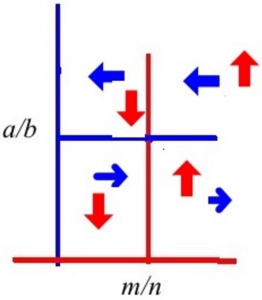
$$J(\tfrac{m}{n}, \tfrac{a}{b}) = \begin{pmatrix} 0 & -\frac{bm}{n} \\ \frac{na}{b} & 0 \end{pmatrix}$$

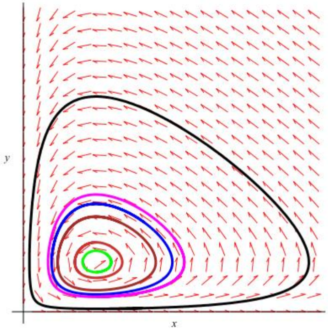
Characteristic Equation: $\lambda^2 + am = 0$ so $\lambda = \pm i\sqrt{am}$

Solutions of Linear System will involve $\sin\sqrt{am}t, \cos\sqrt{am}t$.

Linear System has center.




a/b
m/n


y
x

More on Classic Lotka–Volterra

Linear System Near $(\frac{m}{n}, \frac{a}{b})$

$$u' = -\frac{bm}{n} v$$
$$v' = \frac{an}{b} u$$

$$\frac{v'}{u'} = \frac{an}{b} u \times \frac{n}{-bm} v = -\frac{an^2}{b^2 m} \frac{u}{v}$$

Separate Variables: $b^2 m\, v\, v' = -an^2 u\, u\, u'$

$$b^2 m v^2 = -an^2 u^2 + C$$
$$an^2 u^2 + b^2 m v^2 = C$$

Orbit is an ellipse.

Solutions are linear combinations of $\sin \sqrt{am} t$ and $\cos \sqrt{am} t$.

These are periodic with average values $\frac{m}{n}$(prey), $\frac{a}{b}$(predator)

A Surprising Result

$x' = ax - bxy, y' = mxy - ny$ has Average Values $x^* = \frac{m}{n}, y^* = \frac{a}{b}$

Suppose predators are birds and prey are mosquitos.

We spray insecticide to decrease further the number of mosquitos.

We now have

$$x' = ax - bxy - cx, y' = mxy - ny - d \text{ with } c > 0, d > 0$$

$$x' = x(a - c - by), y' = y(mx - (n + d)).$$

New Equilibrium is $x^* = \dfrac{n+d}{m}, y^* = \dfrac{a-c}{b}$

WE INCREASE THE AVERAGE NUMBER OF MOSQUITOS WHILE DECREASING THE AVERAGE NUMBER OF BIRDS!

Exact Orbits For Classic Lotka-Volterra

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{y(mx-n)}{x(a-by)} = \frac{y}{a-by} \frac{mx-n}{x}$$

$$\frac{dy}{dx} = \left(\frac{y}{a-by} \right) \left(\frac{mx-n}{x} \right)$$

Separate Variables and Integrate

$$\int \left(\frac{a-by}{y} \right) \, dy = \int \left(\frac{mx-n}{x} \right) \, dx$$

$$\int \left(\frac{a - by}{y}\right)\,dy = \int \left(\frac{mx - n}{x}\right)\,dx$$

$$\int \frac{a}{y} - b\,dy = \int m - \frac{n}{x}\,dx$$

$$a \ln y - by = mx - n \ln x + C$$

$$\ln y^a - by = mx - \ln x^n + C$$

Exponentiate each side:

$$e^{\ln y^a - by} = e^{mx - \ln x^n + C}$$

$$e^{\ln y^a} e^{-by} = e^{mx} e^{-\ln x^n} e^C$$

$$y^a e^{-by} = e^{mx} x^{-n} K$$

$$\left(y^a e^{-by}\right)\left(x^n e^{-mx}\right) = K$$

Let $u = y^a e^{-by}$ and $v = x^n e^{-mx}$

Then $uv = K$

We can graph

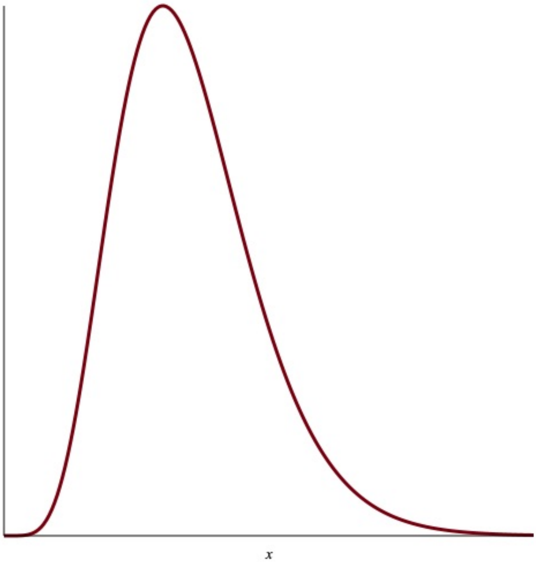
- $uv = K$ in a (u, v)-plane
- $v = x^n e^{-mx}$ in a (x, v)-plane
- $u = y^a e^{-by}$ in a (y, u)-plane

$$
\begin{aligned}
f(x) &= x^n e^{-mx} \\
f'(x) &= x^{n-1} e^{-mx} (n - mx) \\
f''(x) &= x^{n-2} e^{-mx} \left(m^2 x^2 - 2mnx + n^2 - n\right)
\end{aligned}
$$

- $f(x) \geq 0$, all x with $f(x) = 0$ if and only if $x = 0$.
- $f'(x) > 0$ if $x < n/m$ and $f'(x) < 0$ for $x > n/m$
  Hence there is a maximum at $x = \frac{n}{m}$.
- $f''(x) < 0$ for $\frac{n-\sqrt{n}}{m} < x < \frac{n+\sqrt{n}}{m}$
  and positive outside this interval. Points of Inflection at $\frac{n \pm \sqrt{n}}{m}$.

Graph of $x^n e^{-mx} = \dfrac{x^n}{e^{mx}}, x \geq 0$ is

- Increasing and concave up on $[0, \frac{n-\sqrt{n}}{m}]$
- Increasing and concave down on $[\frac{n-\sqrt{n}}{m}, \frac{n}{m}]$.
- Decreasing and concave down on $[\frac{n}{m}, \frac{n+\sqrt{n}}{m}]$,
- Decreasing and concave up on $[\frac{n+\sqrt{n}}{m}, \infty)$.
- $\lim_{x\to\infty} f(x) = 0$ (Repeated Use of l'Hôpital's Rule).


x

Examples of Predator – Prey With Logistic Prey

$$a = 1, p = 1/2, b = 1/2, m = 1/4, n = 1/2$$
$$(x^*, y^*) = \left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

$$J\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & 0 \end{pmatrix}$$

Characteristic Polynomial: $\lambda^2 + \frac{1}{4}\lambda + \frac{3}{16}$

Eigenvalues: $\lambda = \frac{-1 \pm i\sqrt{11}}{8}$

Examples of Predator – Prey With Logistic Prey

$$a = 16, p = 5/2, b = 7/8, m = 10, n = 2$$
$$(x^*, y^*) = (5, 4)$$

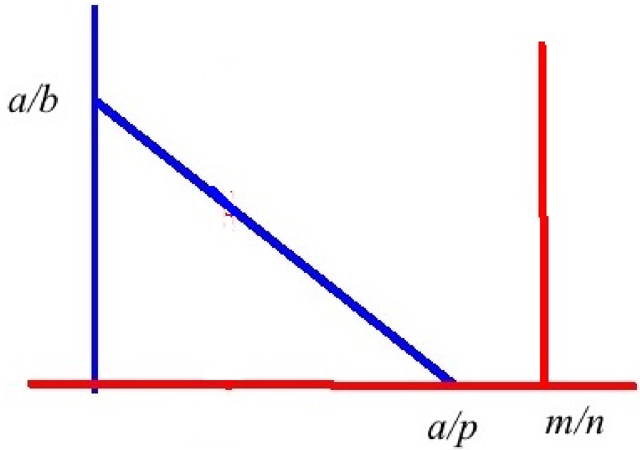
$$J(5,4) = \begin{pmatrix} -\frac{25}{2} & -\frac{35}{8} \\ 8 & 0 \end{pmatrix}$$

Characteristic Polynomial: $\lambda^2 + \frac{25}{2}\lambda + 35$

Eigenvalues: $\lambda = \frac{-25 \pm \sqrt{65}}{4}$

Both eigenvalues are negative.

The Remaining Case



$$\frac{a}{p} < \frac{m}{n}$$

The Fragility of Being a Center

Consider $\mathbf{X'} = A\mathbf{X}$ with $A = \begin{pmatrix} 36 & 80 \\ -50 & -36 \end{pmatrix}$

Characteristic Polynomial: $\lambda^2 + 2704$ so eigenvalues are $\lambda = \pm 52i$

Suppose we replace 36 with $36 + \epsilon$ where ϵ is a small number.

$$A = \begin{pmatrix} 36+\epsilon & 80 \\ -50 & -36 \end{pmatrix}$$

Characteristic Polynomial: $\lambda^2 - \epsilon\lambda + 2704 - 36\epsilon$ so eigenvalues are

$$\lambda = \frac{\epsilon \pm \sqrt{\epsilon^2 + 144\epsilon - 10816}}{2}$$

ϵ small **positive** means real part $\frac{\epsilon}{2} > 0$: Spiral Source

ϵ small **negative** means real part $\frac{\epsilon}{2} < 0$: Spiral Sink

Poincaré – Bendixson Theorem

Henri Poincaré
1854 – 1912

Ivar Bendixson
1861 –1935