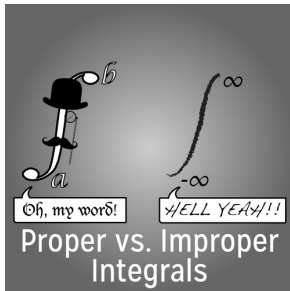


MATH 223: Multivariable Calculus



Class 26: April 20, 2022



Notes on Assignment 24
Assignment 25

Improper Integrals and Probability Density Functions

Announcements

**Today: Improper Integrals
Applications in Probability and Statistics**

Friday: Student Symposium

Monday: Line Integrals

Wednesday: Weighted Curves and Surfaces of Revolution

Improper Integrals

Setting $\int_{\mathcal{B}} f \, dV$ where \mathcal{B} is a subset of \mathbb{R}^n and $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

Two Types:

(I): \mathcal{B} is unbounded

(II) \mathcal{B} is bounded but f is unbounded

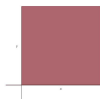
Type I Examples

$\mathcal{B} = \mathbb{R}^2$



$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx$$
$$\int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} f^*(r, \theta) r \, d\theta \, dr$$

$\mathcal{B} = \text{First Quadrant}$



$$\int_0^{\infty} \int_0^{\infty} f(x, y) \, dy \, dx$$
$$\int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} f^*(r, \theta) r \, d\theta \, dr$$

\mathcal{B} is infinite strip



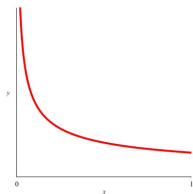
$$\int_{-1}^{\infty} \int_1^2 f(x, y) \, dy \, dx$$

$$\int_{-1}^{\infty} \int_1^2 f(x, y) \, dy \, dx = \lim_{b \rightarrow \infty} \int_{-1}^b \int_1^2 f(x, y) \, dy \, dx$$

Type II Examples

Classic Case

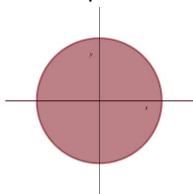
$$I = \int_0^1 \frac{1}{\sqrt{x}} dx$$



$$I = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} [2\sqrt{x}]_a^1 = \lim_{a \rightarrow 0^+} [2 - 2\sqrt{a}] = 2$$

Type II Examples

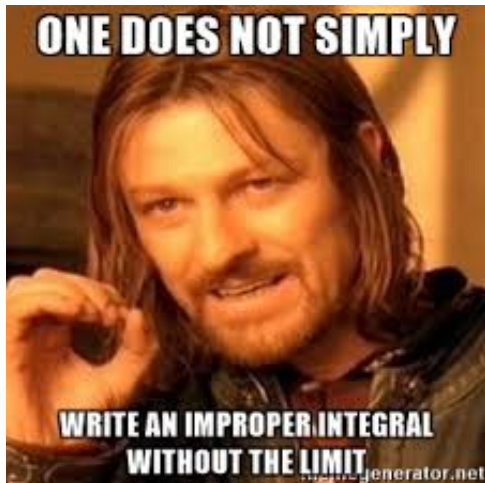
In \mathbb{R}^2 , $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$ on unit disk



In Polar Coordinates:

$$\begin{aligned} \int_0^1 \int_0^{2\pi} \frac{1}{r} r \, d\theta \, dr &= \lim_{a \rightarrow 0^+} \int_a^1 \int_0^{2\pi} d\theta \, dr = \lim_{a \rightarrow 0^+} \int_a^1 2\pi \, dr \\ &= \lim_{a \rightarrow 0^+} (2\pi - 2\pi a) = 2\pi \end{aligned}$$

ONE DOES NOT SIMPLY



**WRITE AN IMPROPER INTEGRAL
WITHOUT THE LIMIT**

generator.net

Improper Integrals

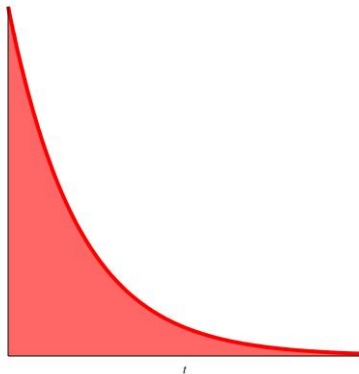
Let $\{B_\delta\}$ be a family of bounded sets B_δ that expands to cover all of the set B . We say $\int_B f(\mathbf{x})dV$ is defined as an **improper integral** if the limit

$$\int_B f(\mathbf{x})dV = \lim_{B_\delta} \int_{B_\delta} f(\mathbf{x}) dV$$
 is finite and independent of the family $\{B_\delta\}$

used to define it. If the limit exists (as a finite number), we say that the improper integral **converges** to that value. If the limit fails to exist, we say the improper integral **diverges**.

An Important Example:
Exponential Probability Density Function

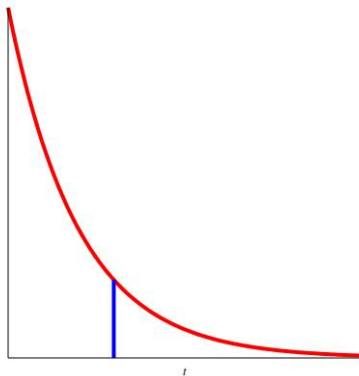
$$\begin{aligned}\int_0^{\infty} e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \Big|_{x=0}^b \right] \\ &= \lim_{b \rightarrow \infty} \left[-e^{-b} - (-e^0) \right] = \lim_{b \rightarrow \infty} \left[1 - \frac{1}{e^b} \right] = 1\end{aligned}$$



Exponential Probability Density Function

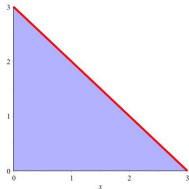
Probability(Light Bulb Burns Out in $\leq x$ months) =

$$\int_0^x e^{-t} dt = 1 - e^{-x}$$



x	$\int_0^x e^{-t} dt$	Prob(Bulb Lasts More than x months)
1	.632	.368
2	.865	.135
3	.950	.050
4	.982	.018

Suppose You Buy 2 Light Bulbs
What Is The Probability They Will Provide At Least 3
Months of Service?



$$\text{Prob}(x + y > 3) = 1 - \text{Prob}(x + y \leq 3)$$

$$= 1 - \int_{x=0}^3 \int_{y=0}^{3-x} e^{-x} e^{-y} dy dx$$

Evaluate $1 - \int_{x=0}^3 \int_{y=0}^{3-x} e^{-x} e^{-y} dy dx$

$$= 1 - \int_0^3 e^{-x} \left[-e^{-y} \Big|_{y=0}^{3-x} \right] dx$$

$$= 1 - \int_0^3 e^{-x} [-e^{3-x} + 1] dx$$

$$= 1 - \int_0^3 (e^{-x} - e^{-3}) dx$$

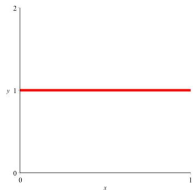
$$= 1 - [-e^{-x} - e^{-3}x]_{x=0}^3$$

$$= 1 - [-e^{-3} - 3e^{-3} + 1 + 0] = 1 - \left[1 - \frac{4}{e^3} \right] = \frac{4}{e^3} \approx .199$$

Probability Density Function

A real-valued function p such that $p(\vec{x}) \geq 0$ for all \vec{x} and $\int_S p = 1$ where S is the set of all possibilities.

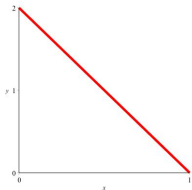
Example 1 Uniform Density: $p(x) = 1$ on $[0,1]$



$$\int_S p = \int_0^1 1 = x \Big|_0^1 = 1$$

Example 2: $p(x) = 2 - 2x$ on $[0,1]$

More likely to choose small numbers than larger numbers



Problem: Find the probability of picking a number less than $1/2$.

$$\int_0^{1/2} (2 - 2x) dx = (2x - x^2) \Big|_0^{1/2} = (1 - \frac{1}{4}) - (0 - 0) = \frac{3}{4}$$

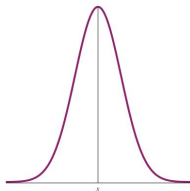
A probability density function on a set S in \mathbb{R}^n is a continuous non-negative real-valued function $p : S \rightarrow \mathbb{R}^1$ such that

$$\int_S p dV = 1$$

If an experiment is performed where S is the set of all possible outcomes, then the probability that the outcome lies in a particular subset T is $\int_T p(\vec{x}) dV$.

Example: **The Bell Curve:** The most important curve in statistics

Start with $y = e^{-\frac{x^2}{2}}$



Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

Point of inflection at $(1, \frac{1}{\sqrt{e}}) = (1, .606)$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$

Impossible to find antiderivative of $e^{-\frac{x^2}{2}}$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$

$$\begin{aligned} A^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right) \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dy dx \end{aligned}$$

Switch To Polar Coordinates: $A^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}} r d\theta dr$

$$A^2 = 2\pi \int_{r=0}^{\infty} re^{-\frac{r^2}{2}} dr = 2\pi \lim_{b \rightarrow \infty} \int_{r=0}^b re^{-\frac{r^2}{2}} dr$$

$$= 2\pi \lim_{b \rightarrow \infty} \left[-e^{-\frac{r^2}{2}} \right]_0^b = 2\pi \lim_{b \rightarrow \infty} \left[-\frac{1}{e^{b^2/2}} + \frac{1}{e^0} \right] = 2\pi \times 1 = 2\pi$$

Thus $A^2 = 2\pi$ so $A = \sqrt{2\pi}$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

To get a probability density, let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
This density is called the **Standard Normal Density**

Example: Suppose two numbers b and c are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing b and c is equivalent to choosing a point (b, c) from the unit square S with $p(\vec{x}) = 1$ (**Uniform Density**)

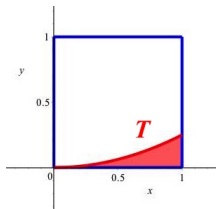
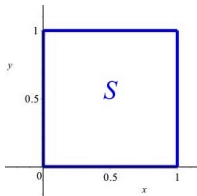
Then $\int_S p(\vec{x}) = \int_S 1 = \text{area}(S) = 1$.

Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Let $T = \{(b, c) : c \leq \frac{b^2}{4}\}$

$$\int_T p(\vec{x}) = \int_{x=0}^1 \int_{y=0}^{x^2/4} 1 \, dy \, dx = \int_{x=0}^1 \frac{x^2}{4} \, dx = \frac{x^3}{12} \Big|_0^1 = \frac{1}{12}$$



General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^{\infty} \lambda e^{-\lambda x} dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^{\infty} \lambda x e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\text{Prob}(\text{Bulb life} \geq 3) = 1 - \int_3^{\infty} \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x} \Big|_3^{\infty} = 1 - e^{-3\lambda}$$

$$\text{Prob}(2 \text{ lights have life} \geq 3) = e^{-3\lambda}(1 + 3\lambda)$$

$$\text{More than } b \text{ hours: } e^{-3b\lambda}(1 + b\lambda)$$