MATH 226: Differential Equations

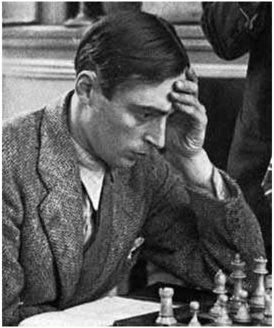
Class 25: April 18, 2022

Assignment 16

Maple Examples in Handouts Folder:
Simple Competition Model
Competition Model

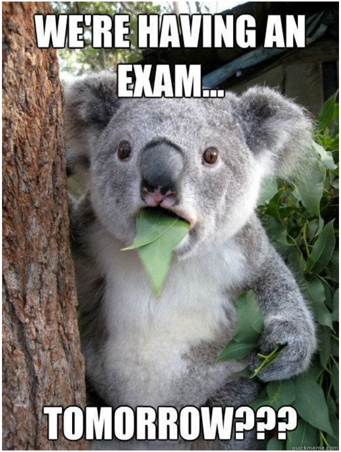
Mathematician of the Week

Conel Hugh Alexander

April 19, 1909 – February 15, 1974
Cryptanalyst and Chess Champion
Biography

G. H. Hardy: "He was the only genuine mathematician I knew who did not become a professional mathematician"



Start at 7 PM in This Room

Schedule

Today	Competition Models
Wednesday	Predator-Prey Models
Friday	Student Symposium

Current Goal:
Approximating Nonlinear Autonomous System with Linear System Near An Equilibrium Point

$$x' = F(x, y)$$
$$y' = G(x, y)$$

$$F(x, y) \approx F(x^*, y^*) + F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G(x^*, y^*) + G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

But at Equilibrium Points, $F(x^*, y^*) = 0, G(x^*, y^*) = 0$ so

$$F(x, y) \approx F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

$$F(x,y) \approx F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

which we can write as

$$\begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix} \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix}$$

or

$$J(x^*,y^*) \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix} = J(x^*,y^*) \begin{bmatrix} h \\ k \end{bmatrix}$$

J is called the **Jacobi Matrix** or **Jacobian**

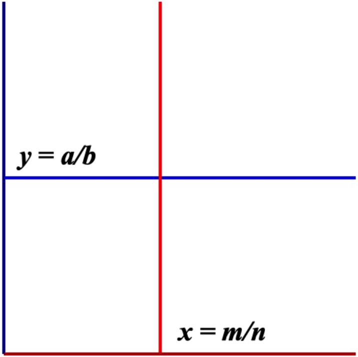
Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x, y)$$
$$y' = my - nxy = y(m - nx) = G(x, y)$$
$$a, b, m, n > 0$$

Search For Equilibrium Points

$$x' = 0 \text{ at } x = 0, y = \frac{a}{b}$$
$$y' = 0 \text{ at } y = 0, x = \frac{m}{n}$$

Critical Points: (0,0) and $(\frac{m}{n}, \frac{a}{b})$: One Point Orbits



Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x,y)$$
$$y' = my - nxy = y(m - nx) = G(x,y)$$
$$a, b, m, n > 0$$

Search For Equilibrium Points

$x' = 0$ at $x = 0, y = \frac{a}{b}$

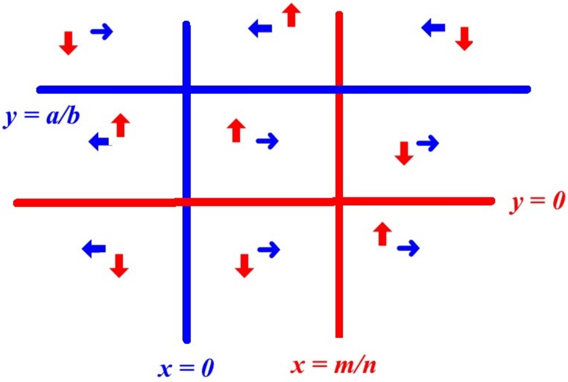
$y' = 0$ at $y = 0, x = \frac{m}{n}$

Critical Points: (0,0) and $(\frac{m}{n}, \frac{a}{b})$: One Point Orbits

x-axis is an orbit and y-axis is an orbit.

$x' > 0$	$y' > 0$
$x(a - by) > 0$ Both Positive $x > 0$ and $y < \frac{a}{b}$ OR Both Negative $x < 0$ and $y > \frac{a}{b}$	$y(m - nx) > 0$ Both Positive $y > 0$ and $x < \frac{m}{n}$ OR Both Negative $y >=< 0$ and $x > \frac{m}{n}$

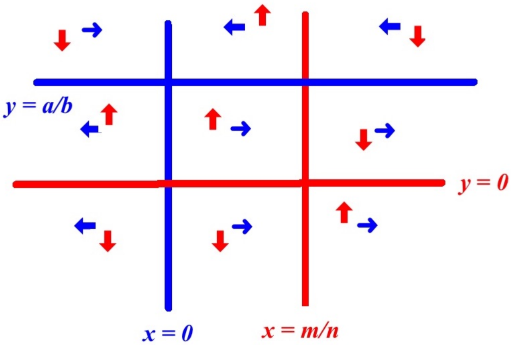
$x' > 0$	$y' > 0$
$x(a - by) > 0$ Both Positive $x > 0$ and $y < \frac{a}{b}$ OR Both Negative $x < 0$ and $y > \frac{a}{b}$	$y(m - nx) > 0$ Both Positive $y > 0$ and $x < \frac{m}{n}$ OR Both Negative $y >=< 0$ and $x > \frac{m}{n}$



Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x, y)$$
$$y' = my - nxy = y(m - nx) = G(x, y)$$
$$a, b, m, n > 0$$
$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$
$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & m \end{bmatrix}$$

Both Eigenvalues are positive

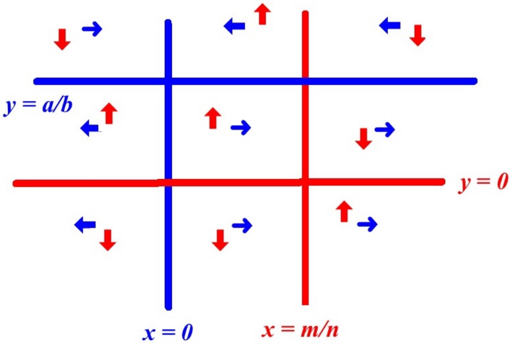


$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$
$$J\left(\tfrac{m}{n}, \tfrac{a}{b}\right) = \begin{bmatrix} a - b\frac{a}{b} & -b\frac{m}{n} \\ -n\frac{a}{b} & m - n\frac{m}{n} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{bm}{n} \\ -\frac{na}{b} & 0 \end{bmatrix}$$

Characteristic Polynomial $\lambda^2 - \frac{-an}{b}\frac{bm}{n} = \lambda^2 - am$

Eigenvalues $\pm\sqrt{am}$ so we have Saddle Point.

Eigenvectors: $\mathbf{v} = \begin{pmatrix} -bm \\ n\sqrt{am} \end{pmatrix}, \mathbf{w} = \begin{pmatrix} bm \\ n\sqrt{am} \end{pmatrix}$



A Specific Example: $a = .9, b = .6, m = .4, n = .3$

Critical Point $(\frac{.4}{.3}, \frac{.9}{.6}) = (\frac{4}{3}, \frac{3}{2})$

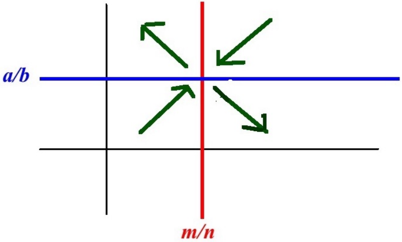
$$J() = \begin{bmatrix} 0 & -\frac{(.6)(.4)}{.3} \\ -\frac{.27}{.6} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{4}{5} \\ -\frac{9}{20} & 0 \end{bmatrix}$$

Eigenvalue	Eigenvector	Solution
$\lambda = \sqrt{36/100} = \frac{3}{5}$	$\mathbf{v} = \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$	$e^{\frac{3}{5}t}\mathbf{v}$
$\lambda = -\sqrt{(36/100} = -\frac{3}{5}$	$\mathbf{w} = \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$	$e^{-\frac{3}{5}t}\mathbf{w}$

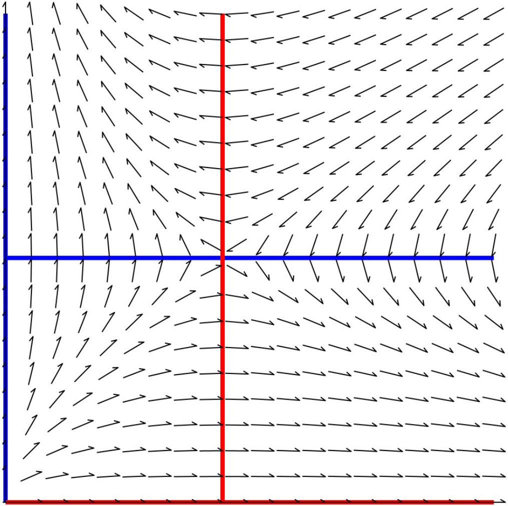
General Solution of Linear System

$$C_1 e^{\frac{3}{5}t} \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} + C_2 e^{-\frac{3}{5}t} \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$$

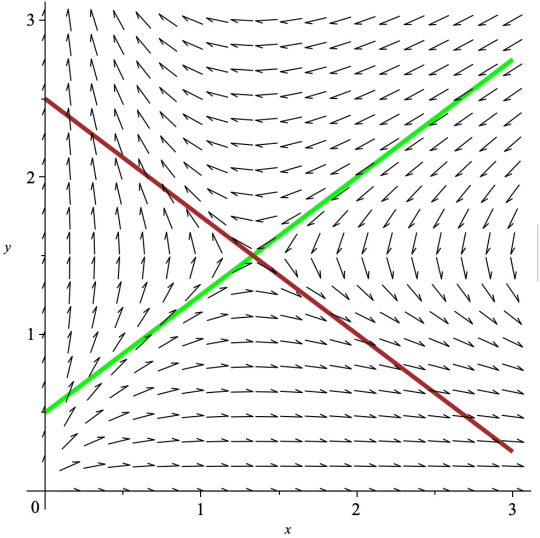
Behavior Near Equilibrium Point



$$C_1 e^{\frac{3}{5}t} \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} + C_2 e^{-\frac{3}{5}t} \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$$



Phase Portrait With Nullclines



Phase Portrait With Eigenvectors

Simple Model $x' = ax - bxy$

$y' = my - nxy$

$a, b, m, n > 0$

Each Species Grows **Exponentially** in Absence of Other.

More Realistic Model

$x' = ax - px^2 - bxy$

$y' = my - qy^2 - nxy$

$a, b, m, n > 0$

Each Spcies Grows **Logistically** in Absence of Other.

Next Time

Predator-Prey Models