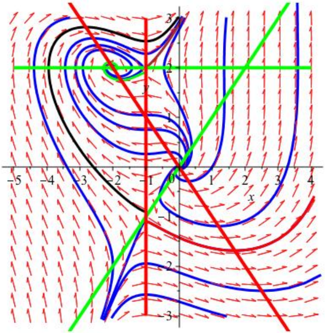
MATH 226: Differential Equations



Class 23: April 13, 2022

Variation of Parameters 1

Variation of Parameters 2

Today's Agenda

- Review Matrix Exponential
- Nonhomogenous Systems
- More on Defective Matrices
- Quick Peek at Nonlinear Systems

Review

The Matrix Exponential

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots + A^k\frac{t^k}{k!} + \ldots$$

e^{At} is an $n \times n$ matrix
Each column of e^{At} is a solution of $\mathbf{x'} = A\mathbf{x}$
The columns form a linearly independent set

Some Other Nice Properties:

$$e^{A\times 0} = I$$
$$(e^{At})' = Ae^{At}$$
$$e^{-At} = (e^{At})^{-1}$$
$$e^{A(r+s)} = e^{Ar}e^{As}$$

Review

e^{At} **has wonderful properties but it is hard to compute via the power series definition.**

Alternate Way To Compute Matrix Exponential e^{At}

$$e^{At} = X(t)(X(0))^{-1}$$

where $X(t)$ is any Fundamental Matrix for x' $=$ Ax.

How To Find X?
Use eigenvalue/eigenvector approach.

Nonhomogeneous Systems

Recall Solution of $x' = ax + g(t)$

$x' - ax = g(t)$

Multiply by integrating factor e^{-at}

$(xe^{-at})' = e^{-at}g(t)$

$$xe^{-at} = \int e^{-at}g(t)\,dt + C$$

$$x = e^{at}\int e^{-at}g(t)\,dt + Ce^{at}$$

$$x = e^{at}\int_0^t e^{-as}g(s)\,ds + Ce^{at}$$

Evaluate at $t = 0$:

$$x = e^{at}\int_0^t e^{-as}g(s)\,ds + x(0)e^{at}$$

Nonhomogeneous Systems

$x' = ax + g(t)$ has solution $x = e^{at}\int_0^t e^{-as} g(s)\, ds + e^{at} x(0)$

$\mathbf{X'} = A\mathbf{X} + \mathbf{g}(t)$ has solution

$$\mathbf{X} = e^{At}\int_0^t e^{-As}\mathbf{g}(s) + e^{At}\mathbf{X}(0)$$

$$\mathbf{X} = \Phi(t)\int_0^t \Phi^{-1}(s)\mathbf{g}(s) + \Phi(t)\mathbf{X}(0)$$

We can also write the solution as

$$\mathbf{X}(t)\int_{t_0}^t \mathbf{X}^{-1}(s)\mathbf{g}(s)\, ds + \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$$

where $\mathbf{X}$ is any fundamental solution of $\mathbf{X'} = A\mathbf{X}$

More on Defective Matrices

Solving **x'** = A**x** when A is "defective"

Suppose λ is an eigenvalue of A with algebraic multiplicity 3 but geometric multiplicity 1.

Find **v** such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Find **w** such that $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Find **u** such that $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Then 3 linearly independent solutions of **x'** = A**x** are

$$e^{\lambda t}\mathbf{v}$$

$$te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$

$$\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$$

What to do if λ is an eigenvalue of A with algebraic multiplicity **bigger than 3** but geometric multiplicity 1?

WHY DOES THIS WORK?

KEY STEP:

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \text{ implies } A\mathbf{v} = \lambda\mathbf{v}$$
$$(A - \lambda I)\mathbf{w} = \mathbf{v} \text{ implies } A\mathbf{w} = \mathbf{v} + \lambda\mathbf{w}$$
$$(A - \lambda I)\mathbf{u} = \mathbf{w} \text{ implies } A\mathbf{u} = \mathbf{w} + \lambda\mathbf{u}$$

Find $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Find $\mathbf{w}$ such that $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Find $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Note: 3 systems of linear algebraic equations with the **same coefficient matrix**

Example $\begin{pmatrix} 4 & 8 & 12 \\ -82 & 204 & 295 \\ 64 & -128 & -184 \end{pmatrix}$

Characteristic Polynomial : $\lambda^3 - 24\lambda^2 + 192\lambda - 512 = (\lambda - 8)^3$

Eigenvalue $\lambda = 8$ has algebraic multiplicity 3, but geometric multiplicity only 1.

$$\text{Here } (A - \lambda I) = \begin{pmatrix} -4 & 8 & 12 \\ -82 & 196 & 295 \\ 64 & -128 & -192 \end{pmatrix}$$

To solve $(A - \lambda I)\mathbf{x} = \mathbf{b}$ construct augmented matrix

$$\left(\begin{array}{ccc|c} -4 & 8 & 12 & a \\ -82 & 196 & 295 & b \\ 64 & -128 & -192 & c \end{array}\right)$$

and reduce to row echelon form

To solve $(A - \lambda I)\mathbf{x} = \mathbf{b}$:

Augmented matrix is $\begin{pmatrix} -4 & 8 & 12 & | \, a \\ -82 & 196 & 295 & | \, b \\ 64 & -128 & -192 & | \, c \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & \frac{1}{16} & | \, \frac{49}{32}a + \frac{1}{16}b \\ 0 & 1 & \frac{49}{32} & | - \frac{41}{64}a + \frac{1}{32}b \\ 0 & 0 & 0 & | \, 16a + c \end{pmatrix} \text{ so } \mathbf{x} = \begin{pmatrix} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ 16a + c \end{pmatrix}$$

For $(A - \lambda I)\mathbf{v} = \mathbf{0}$, set $a = 0, b = 0, c = 0$:

$$\mathbf{v} = \begin{pmatrix} -(1/16)x_3 \\ -(49/32)x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1/16 \\ -49/32 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -49 \\ 32 \end{pmatrix} \text{ if } x_3 = 1$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{16} & \mid \frac{49}{32}a + \frac{1}{16}b \\ 0 & 1 & \frac{49}{32} & \mid -\frac{41}{64}a + \frac{1}{32}b \\ 0 & 0 & 0 & \mid 16a + c \end{pmatrix} \text{ so } \mathbf{x} = \begin{pmatrix} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ 16a + c \end{pmatrix}$$

For $(A - \lambda I)\mathbf{w} = \mathbf{v}$, set $a = -2, b = -49, c = 32$:

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} (-1/16)w_3 \\ 1/4 - (49/32)w_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ 0 \end{pmatrix} \text{ if } w_3 = 0.$$

$$\mathbf{x} = \begin{pmatrix} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ \\ 16a + c \end{pmatrix}$$

For $(A - \lambda I)\mathbf{u} = \mathbf{w}$, set $a = 0, b = -1/4, c = 0$:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (-1/64) - (1/16)u_3 \\ (-1/128) - (49/32)u_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} -1/64 \\ -1/128 \\ 0 \end{pmatrix} \text{ if } u_3 = 0.$$

$$\text{Thus } \mathbf{v} = \begin{pmatrix} -2 \\ -49 \\ 32 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1/4 \\ 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -1/64 \\ -1/128 \\ 0 \end{pmatrix}$$

Solving **x'** = A**x** when A is "defective"

Suppose λ is an eigenvalue of A with algebraic multiplicity 4 but geometric multiplicity 1.

Find **v** such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$
Find **w** such that $(A - \lambda I)\mathbf{w} = \mathbf{v}$
Find **u** such that $(A - \lambda I)\mathbf{u} = \mathbf{w}$
Find **z** such that $(A - \lambda I)\mathbf{z} = \mathbf{u}$

Then 4 linearly independent solutions of **x'** = A**x** are

$$e^{\lambda t}\mathbf{v}$$

$$te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$

$$\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$$

$$\frac{t^3}{3!}e^{\lambda t}\mathbf{v} + \frac{t^2}{2}e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{u} + e^{\lambda t}\mathbf{z}$$

Next Major Goal:
Study Nonlinear Systems of General First Order Differential Equations

$$\begin{aligned} x' &= F(x, y, t) \\ y' &= G(x, y, t) \end{aligned} \qquad \begin{aligned} x' &= F(x, y, z, t) \\ y' &= G(x, y, z, t) \\ z' &= H(x, y, z, t) \end{aligned}$$

General Case

$$\begin{aligned} x_1' &= f_1(x_1, x_2, ..., x_n, t) \\ x_2' &= f_2(x_1, x_2, ..., x_n, t) \\ &\vdots \\ x_n' &= f_n(x_1, x_2, ..., x_n, t) \end{aligned}$$

$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

Autonomous Systems

No Explicit t on Right Hand Side

$n = 2$	$n = 3$
$x' = F(x, y)$	$x' = F(x, y, z)$
$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$

Three Approaches:

Analytic: Rarely Possible To Find Closed Form Solution

Numeric: Detailed Information About a Single Solution

Geometric: Qualitative Information About All Solutions

Example

$$x' = (2 - y)(x - y)$$
$$y' = (1 + x)(x + y)$$

STEP ONE: Identify All Equilibrium Points

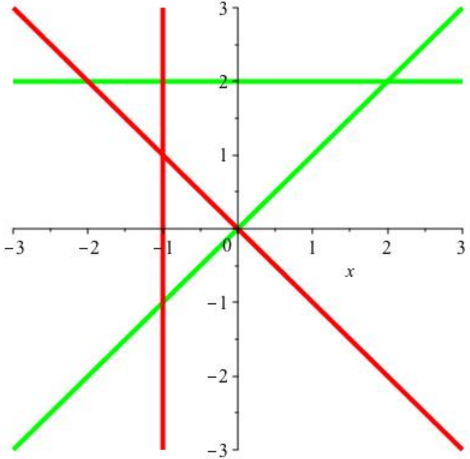
x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x

$$x' = (2-y)(x-y), y' = (1+x)(x+y)$$

x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x



$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

Autonomous Systems

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$n = 2$	$n = 3$
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$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$

Special Properties of Autonomous Systems

1. Direction Field is Independent of Time
2. Only One Trajectory Passing Through Each Point (x_0, y_0)
3. A Trajectory Can Not Cross Itself
4. A Single Well-Chosen Phase Portrait Simultaneously Displays Important Information About All Solutions

Analyzing a Nonlinear System

Our Example From Last Time:

$$x' = F(x, y) = (2 - y)(x - y) = 2x - 2y - xy + y^2$$
$$y' = G(x, y) = (1 + x)(x + y) = x + y + x^2 + xy$$

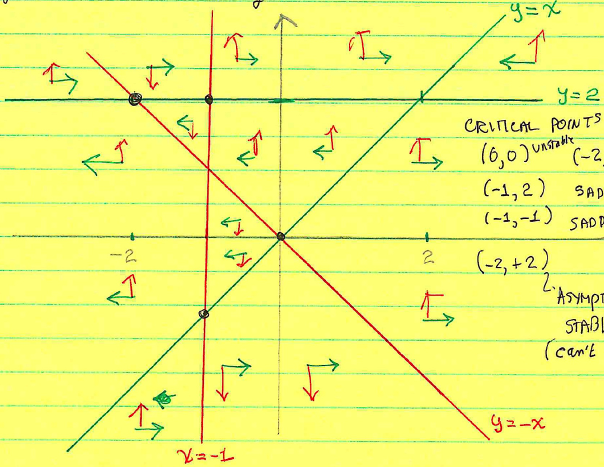
Step 1: Find Critical Points: $x' = 0$ and $y' = 0$
Step 2: Find Signs of x' and y'

EXAMPLE $x' = (2-y)(x-y) = 2x - 2y - xy + y^2$

$y' = (1+x)(x+y) = x + y + x^2 + xy$

$x' = 0$ at $y = 2$ and $y = x$ GREEN

$y' = 0$ at $x = -1$ and $y = -x$



CRITICAL POINTS

(0,0) unstable

(-2, 2)

(-1, 2) SADDLE

(-1, -1) SADDLE

(-2, +2)

2. ASYMPTOTICALLY STABLE (can't tell)

Current Goal:
Approximating Nonlinear Autonomous System with Linear System Near An Equilibrium Point

$$x' = F(x,y)$$
$$y' = G(x,y)$$

$$F(x,y) \approx F(x^*,y^*) + F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G(x^*,y^*) + G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

But at Equilibrium Points, $F(x^*,y^*) = 0, G(x^*,y^*) = 0$ so

$$F(x,y) \approx F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

Carl Gustav Jacob Jacobi
December 10, 1804 – February 18, 1851

$$F(x,y) \approx F_x(x^*,y^*)(x-x^*) + F_y(x^*,y^*)(y-y^*)$$
$$G(x,y) \approx G_x(x^*,y^*)(x-x^*) + G_y(x^*,y^*)(y-y^*)$$

which we can write as

$$\begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix} \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix}$$

or

$$J(x^*,y^*) \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix} = J(x^*,y^*) \begin{bmatrix} h \\ k \end{bmatrix}$$

J is called the **Jacobi Matrix** or **Jacobian**