

MATH 226: Differential Equations

DIFFERENTIAL EQUATION

HOW TO FIND

e^{At}

$$e^{At} = 1 + \frac{At}{1!} + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots + \frac{(At)^n}{n!}$$
$$= \sum_{i=0}^n \frac{1}{i!} A^i t^i$$

Class 22: April 11, 2022



Notes on Assignment 13

Assignment 15

Sample Exam 2

Notes on Sample Exam 2

Matrix Exponential Power Series

Computing Matrix Exponential From a Fundamental Matrix

Announcements

Exam 2 Next Monday

Major Goal:

Study Systems of First Order Linear Differential Equations

$$\mathbf{x}' = \mathbf{P}(t) \mathbf{x} + \mathbf{g}(t)$$

where $\mathbf{P}(t)$ is a square matrix of continuous functions
and $\mathbf{g}(t)$ is a vector of continuous functions.

Special Case: $\mathbf{x}' = \mathbf{A} \mathbf{x}$

where \mathbf{A} is a square matrix of constants.

For each eigenvector \mathbf{v} associated with an eigenvalue λ of the
matrix \mathbf{A} :

$$e^{\lambda t} \mathbf{v}$$

is a solution.

Matrix Exponential Function

Our very first example in the course

$$x' = ax \text{ where } a \text{ is a constant}$$

has a solution of the form $x = e^{at}$

$$\text{By analogy, } \mathbf{X}' = \mathbf{A} \mathbf{X}$$

"ought" to have a solution of the form

$$\mathbf{X} = e^{\mathbf{A}t}$$

But What is the Exponential of a Matrix?

Applying Exponential Function to a Matrix

Recall that the square of a matrix is not the matrix of squares.

So we don't expect to get the matrix of exponentials.

Begin with the expression for e^{at} as a power series:

$$e^{at} = 1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \frac{a^4 t^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!}$$

This series converges absolutely for all a and t .

Let's define e^{At} as

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

We can compute each term in this series; it will be an $n \times n$ matrix if A is an $n \times n$ matrix.

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Some Properties of e^{At} :

1. $e^{A0} = e^0 = I$
2. $(e^{At})' = 0 + A + \frac{A^2}{2!} 2t + \frac{A^3}{3!} 3t^2 + \frac{A^4}{4!} 4t^3 \dots +$
 $= A + A^2 t + \frac{A^3}{2!} t^2 + \frac{A^4}{3!} t^3 + \dots$
 $= A(I + At + \frac{A^2}{2!} t^2 + \frac{A^3}{3!} t^3 + \dots)$
 $= Ae^{At}$
 so e^{At} is a solution of $\mathbf{x}' = A\mathbf{x}$.
3. Each column is a solution
4. Columns are linearly independent
5. $e^{-At} = (e^{At})^{-1}$ (Matrix Inverse)
6. $e^{A(s+t)} = e^{As} e^{At}$
7. $e^{(A+B)t} = e^{At} e^{Bt}$ if $AB = BA$
8. $Ae^{At} = e^{At} A$

Computing e^{At} via Power Series

Need A, A^2, A^3, A^4, \dots

Note: $A^2 = AA, A^3 = AA^2, A^4 = AA^3, \text{ etc}$

Example $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$$

$$A^6 = \begin{pmatrix} 5743 & 8370 \\ 12555 & 18298 \end{pmatrix}$$

Not so easy to see what e^{At} actually looks like!

Try a different example: $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

[See *Maple* "Matrix Exponential Power Series"]

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -(a+2c) & -(b+2d) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}, A^3 = \begin{pmatrix} 2 & 3 \\ -3 & -4 \end{pmatrix}, A^4 = \begin{pmatrix} -3 & -4 \\ 4 & 5 \end{pmatrix}$$

A^k

k even

k odd

$$\begin{pmatrix} -(k-1) & -k \\ k & k+1 \end{pmatrix} \quad \begin{pmatrix} k-1 & k \\ -k & -(k+1) \end{pmatrix}$$

General Formula: $A^k = \begin{pmatrix} (-1)^{k+1}(k-1) & (-1)^{k+1}k \\ (-1)^k k & (-1)^k(k+1) \end{pmatrix}$

Our Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

Found: $A^k = \begin{pmatrix} (-1)^{k+1}(k-1) & (-1)^{k+1}k \\ (-1)^k k & (-1)^k(k+1) \end{pmatrix}$

Upper Left: 0, -1, 2, -3, 4, -5, . . .

Upper Right: 1, -2, 3, -4, 5, . . .

Lower Left: -1, 2, -3, 4, -5, . . .

Lower Right: -2, 3, -4, 5, . . .

Power Series:

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Examine Upper Right Entries:

$$\begin{aligned} & 0 + 1t - 2\frac{t^2}{2!} + 3\frac{t^3}{3!} - 4\frac{t^4}{4!} + 5\frac{t^5}{5!} - 6\frac{t^6}{6!} + \dots \\ & = t \left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots \right] = te^{-t} \end{aligned}$$

Work out other three entries:

$$e^{At} = \begin{pmatrix} e^{-t}(t+1) & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{pmatrix}$$

THERE'S GOT TO BE A BETTER WAY!

An Alternative Way To Compute e^{At}

Idea: Let Φ be any fundamental solution matrix for $\mathbf{x}' = A\mathbf{x}$ with

$$\Phi(0) = I.$$

Then $\Phi(t)$ and e^{At} are both solutions of $\mathbf{x}' = A\mathbf{x}$ which satisfy the same initial condition.

The Uniqueness of Solutions Theorem implies $\Phi(t) \equiv e^{At}$

How to find Φ .

1. Use Eigenvalue/ Eigenvector approach to find a full linearly independent set of solutions to $X' = Ax$.
2. Enter them as columns in a matrix in a matrix $X(t)$ (This is a fundamental matrix)
3. $X(t)$ is invertible for all t . Thus $X(0)$ is an invertible matrix of constants
4. Define $\Phi(t) = X(t) [X(0)]^{-1}$

$$\text{Then } \Phi(0) = X(0) [X(0)]^{-1} = I$$

$$\text{and } \Phi'(t) = X'(t) [X(0)]^{-1} = AX(t) [X(0)]^{-1} = A\Phi(t)$$

so Φ is also a solution.

Using this approach for $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -2 - \lambda \end{pmatrix} = 2\lambda + \lambda^2 + 1 = (\lambda + 1)^2$$

$\lambda = -1$ is double root; algebraic multiplicity = 2

To find eigenvectors: $(A - \lambda I) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ which reduces to

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ so } \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Geometric Multiplicity = 1.

One solution is $e^{-1t}\mathbf{v} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

To find another, solve $(A - \lambda I)\mathbf{w} = \mathbf{v}$ for \mathbf{w}

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ implies } \begin{cases} w_1 + w_2 = 1 \\ -w_1 - w_2 = -1 \end{cases}$$

Choose $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solutions are $e^{-t}\mathbf{v}$, $te^{-t}\mathbf{v} + e^{-t}\mathbf{w}$

$$\mathbf{x}_1 = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$\mathbf{x}_2 = te^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} te^{-t} + e^{-t} \\ -te^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t}(t+1) \\ e^{-t}(-t) \end{pmatrix}$$

$$\mathbf{X}(t) = \begin{pmatrix} e^{-t} & e^{-t}(t+1) \\ -e^{-t} & e^{-t}(-t) \end{pmatrix} \text{ so } \mathbf{X}(0) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\text{Then } [\mathbf{X}(0)]^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Thus

$$\begin{aligned} e^{At} = \Phi(t) &= \mathbf{X}(t)[\mathbf{X}(0)]^{-1} = \begin{pmatrix} e^{-t} & e^{-t}(t+1) \\ -e^{-t} & e^{-t}(-t) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-t}(t+1) & te^{-t} \\ -e^{-t}t & e^{-t}(1-t) \end{pmatrix} \end{aligned}$$

Review

The Matrix Exponential

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots + A^k \frac{t^k}{k!} + \dots$$

e^{At} is an $n \times n$ matrix

Each column of e^{At} is a solution of $\mathbf{x}' = A\mathbf{x}$

The columns form a linearly independent set

Some Other Nice Properties:

$$e^{A \times 0} = I$$

$$(e^{At})' = Ae^{At}$$

$$e^{-At} = (e^{At})^{-1}$$

$$e^{A(r+s)} = e^{Ar} e^{As}$$

Review

e^{At} has wonderful properties but it is hard to compute via the power series definition.

Alternate Way To Compute Matrix Exponential e^{At}

$$e^{At} = X(t)(X(0))^{-1}$$

where $X(t)$ is any Fundamental Matrix for $x' = Ax$.

How To Find X ?

Use eigenvalue/eigenvector approach.

Nonhomogeneous Systems

Recall Solution of $x' = ax + g(t)$

$$x' - ax = g(t)$$

Multiply by integrating factor e^{-at}

$$(xe^{-at})' = e^{-at}g(t)$$

$$xe^{-at} = \int e^{-at}g(t) dt + C$$

$$x = e^{at} \int e^{-at}g(t) dt + Ce^{at}$$

$$x = e^{at} \int_0^t e^{-as}g(s) ds + Ce^{at}$$

Evaluate at $t = 0$:

$$x = e^{at} \int_0^t e^{-as}g(s) ds + x(0)e^{at}$$

Nonhomogeneous Systems

$x' = ax + g(t)$ has solution $x = e^{at} \int_0^t e^{-as} g(s) ds + e^{at} x(0)$

$\mathbf{X}' = A\mathbf{X} + \mathbf{g}(t)$ has solution

$$\mathbf{X} = e^{At} \int_0^t e^{-As} \mathbf{g}(s) ds + e^{At} \mathbf{X}(0)$$

$$\mathbf{X} = \Phi(t) \int_0^t \Phi^{-1}(s) \mathbf{g}(s) ds + \Phi(t) \mathbf{X}(0)$$

We can also write the solution as

$$\mathbf{X}(t) \int_{t_0}^t \mathbf{X}^{-1}(s) \mathbf{g}(s) ds + \mathbf{X}(t) \mathbf{X}^{-1}(t_0)$$

where \mathbf{X} is any fundamental solution of $\mathbf{X}' = A\mathbf{X}$