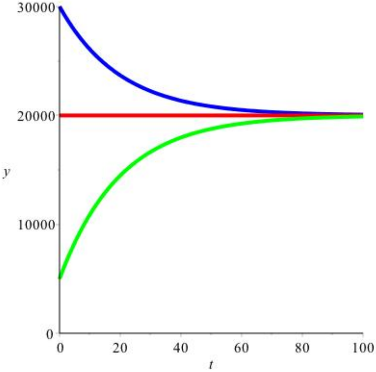
MATH 226: Differential Equations



February 16, 2022

Handouts

Assignment 1 (Integration Review)

Announcements

1. Office Hour Today: 12:30 – 1:30
2. Explore Course Website
3. Practice Problems vs. Feedback Problems

Key Terms From Chapter 1

Independent Variable
Dependent Variable
Parameter
Solution
Equilibrium Solution
Integral Curves
Autonomous Differential Equation
Critical Point = Fixed Point = Stationary Point
Phase Line
One - Dimensional Phase Portrait
Asymptotically Stable
Unstable
Semistable Attractor = Sink
Repeller = Source
Linearization About An Equilibrium
Direction Field

VERIFYING A PROPOSED SOLUTION

Example: Verify that $y = Ce^{-4t} + \frac{1}{4}t + \frac{1}{2}e^{-2t} - \frac{1}{16}$ is a solution of

$$y' + 4y = t + e^{-2t}$$

Verification

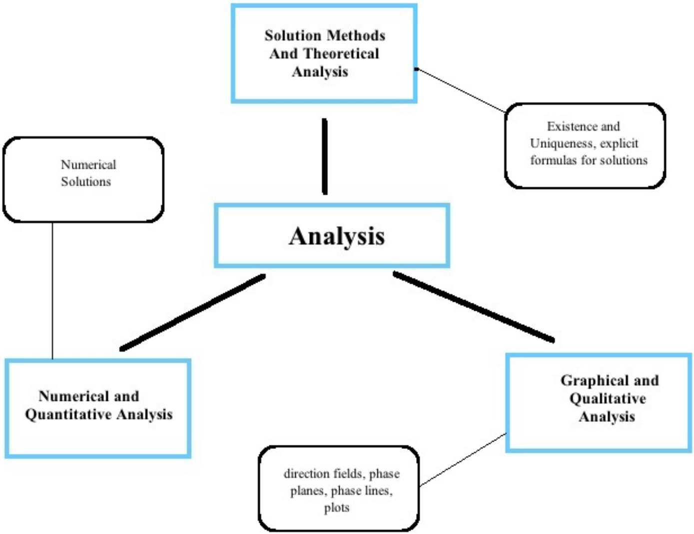
$$y' = -4Ce^{-4t} + \frac{1}{4} - e^{-2t} - 0$$

$$4y = 4Ce^{-4t} + t + 2e^{-2t} - \frac{1}{4}$$

Adding:

$$y' + 4y = t + e^{-2t}$$

Analyzing Differential Equations



Example

$$P'(t) = 3P(t) \text{ with } P(0) = 100$$

Initial Value Problem

Applications:

Colony of Bacteria

Money Compounded Continuously

Human Population with Constant Per Capita Growth Rate

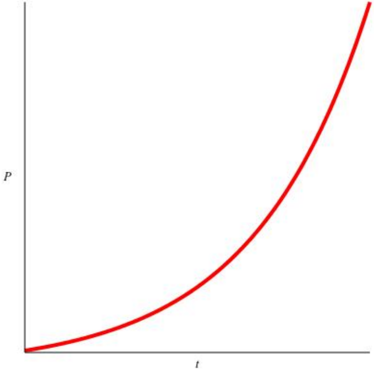
$$P'(t) = 3P(t) \text{ with } P(0) = 100$$

Qualitative Analysis

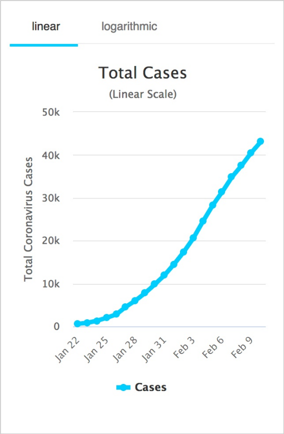
P' is positive so P is increasing.

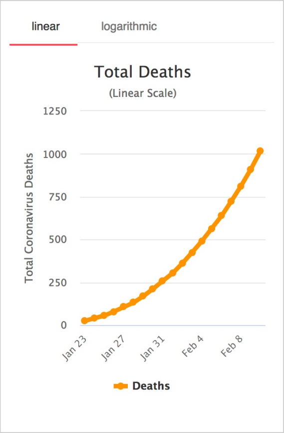
$$P'' = (P')' = (3P)' = 3P' = 3 \times 3P = 9P$$

So $P'' > 0$ and hence graph of P is increasing and concave up.



Do Such Curves Exist in the Real World?





Coronavirus Data

$P'(t) = 3P(t)$ with $P(0) = 100$

Analytic Solution

$\frac{1}{P(t)}P'(t) = 3$

Integrate each side with respect to t

$ln|P(t)| = 3t + C$

But $P(t) > 0$ so

$\ln P(t) = 3t + C$

Apply exponential function to each side:

$e^{lnP(t)} = e^{3t+C} = e^{3t}e^{C} = Ce^{3t}$

Hence $P(t) = Ce^{3t}$

Evaluate at 0:

$100 = P(0) = Ce^{3\times 0} = Ce^{0} = C$

so $P(t) = 100e^{3t}$

P is unbounded

$$\lim_{t\to\infty} P(t) = +\infty$$

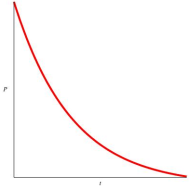
How about $P' = -3P, P(0) = 100$?

Application: Radioactive Decay

P' is initially negative, so P is decreasing

$$P'' = (P')' = (-3P)' = -3P' = -3(-3)P = 9P > 0$$

Hence P is decreasing and graph is concave up



Analytic Solution (Go Through Similar Steps)

$$P(t) = 100e^{-3t}$$

Observe: $P(t) > 0$ for all t

$$\lim_{t \to \infty} P(t) = 0$$

MORE GENERALLY:

$P'(t) = kP(t)$ with $P(0) = P_0$

Analytic Solution

$\frac{1}{P(t)}P'(t) = k$

Integrate each side with respect to t

$ln|P(t)| = kt + C$

But $P(t) > 0$ so

$\ln P(t) = kt + C$

Apply exponential function to each side:

$e^{lnP(t)} = e^{kt+C} = e^{kt}e^{C} = Ce^{kt}$

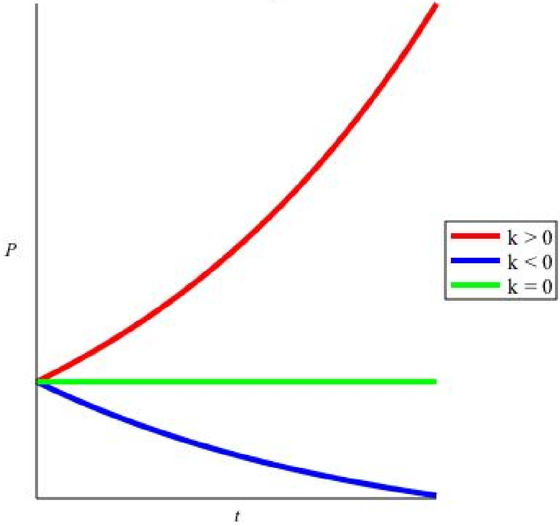
Hence $P(t) = Ce^{kt}$

Evaluate at 0:

$100 = P(0) = Ce^{k \times 0} = Ce^{0} = C$

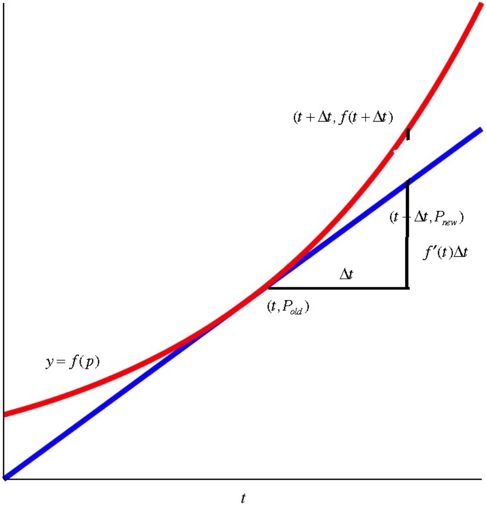
so $P(t) = 100e^{kt}$

Plot of $Po\,e^{kt}$



Euler's Method For "Solving" Numerically $P'(t) = kP(t)$

$$P_{new} = P_{old} + k * P_{old} * \Delta t$$



Euler's Method For "Solving" Numerically $P'(t) = kP(t)$

$$P_{new} = P_{old} + k * P_{old} * \Delta t$$

Example: $k = .04, P(0) = 1000$

Time	Approximate	Exact
0.	1000.00	1000.00
0.1000000000	1004.00000	1004.008011
0.2000000000	1008.016000	1008.032086
0.3000000000	1012.048064	1012.072289
0.4000000000	1016.096256	1016.128685
0.5000000000	1020.160641	1020.201340
0.6000000000	1024.241284	1024.290318
0.7000000000	1028.338249	1028.395684
0.8000000000	1032.451602	1032.517505
0.9000000000	1036.581408	1036.655846
1.000000000	1040.727734	1040.810774

The Differential Equation

$$P'(t) = kP(t)$$

Has Solution (if k is a constant)

$$P(t) = P(0)e^{kt}$$

We can write $P'(t) = kP(t)$ as $P' = kP$
We change the name of the dependent variable:

$$y' = ky \Rightarrow y = y(0)e^{kt}$$
$$x' = kx \Rightarrow x = x(0)e^{kt}$$

More Complicated Relationships

1. Population with immigration and/or emigration
2. Forest Management
3. Fishery Management
4. Lake Champlain Pollution
5. Anesthetic
6. Alcohol/Drug
7. Credit Card Debt

$$P' = aP + b$$

where a and b are constants

$$P' = aP + b \text{ or } y' = ay + b$$

where a and b are constants