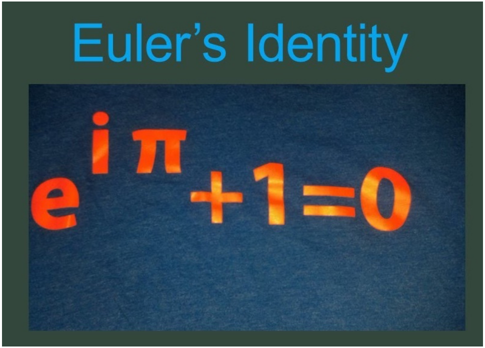
MATH 226: Differential Equations



Class 17: March 30, 2022

Notes on MATLAB Worksheet Two Procedure For Complex Eigenvalues
Some Power Series Representations

Current Goal:

Continue Study of Linear Homogeneous Systems With Constant Coefficients

X' $=$ A X

2 $\times$ 2 Case

Theorem: If λ and μ are distinct eigenvalues (real or complex) of a 2×2 matrix A having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x}' = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

So Far:

- A has unequal real roots (Sources, Sinks, Saddle Points)
- Complex Eigenvalues and Eigenvectors

Consider the system of first order linear homogeneous differential equations

$$x'(t) = 2x(t) + py(t)$$
$$y'(t) = -1x(t) + 3y(t)$$

where p is any real number.

Then for any initial condition $x(0) = x_0, y(0) = y_0$, there is a unique solution of the system $x = f(t), y = g(t)$ satisfying the initial condition.

The values of $f(t)$ and $g(t)$ will be **real** numbers for all t.

Apply To System of Differential Equations

$$X' = AX \text{ with } A = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & 3 \end{pmatrix}$$

We have

$$\lambda = \frac{5+3i}{2} \quad \text{so} \quad \mu = \frac{5-3i}{2}$$

$$\vec{v} = \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 1+3i \\ 2 \end{pmatrix}$$

Solutions of Differential Equations Should be

$$e^{\left(\frac{5+3i}{2}\right)t} \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} \text{ and } e^{\left(\frac{5-3i}{2}\right)t} \begin{pmatrix} 1+3i \\ 2 \end{pmatrix}$$

How Can We Make Sense of

$$e^{\left(\frac{5+3i}{2}\right)t} = e^{\left(\frac{5}{2}t + \frac{3i}{2}t\right)}?$$

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By Rules of Exponents, We Should Have

$$e^{\left(\frac{5}{2}t+\frac{3i}{2}t\right)} = e^{\frac{5}{2}t}e^{\frac{3}{2}it}$$

Euler's Formula:

$$e^{bi} = \cos b + i \sin b$$

so $e^{\frac{3}{2}it} = \cos \frac{3}{2}t + i \sin \frac{3}{2}t$

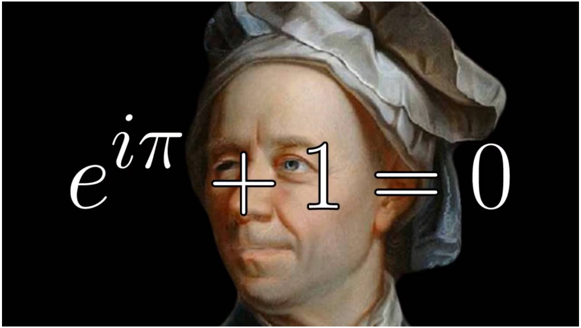
Euler's Formula:

$$e^{bi} = \cos b + i \sin b$$

Note: If $b = \pi$, then

$$e^{\pi i} = \cos \pi + i \sin \pi = -1 + i0 = -1$$

$$\boxed{e^{\pi i} + 1 = 0}$$


$e^{i\pi} + 1 = 0$

Thus

$$e^{\frac{5}{2}t+\frac{3}{2}it}\begin{pmatrix}1-3i\\2\end{pmatrix} = e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\begin{pmatrix}1-3i\\2\end{pmatrix}$$

$$= e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\left[\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-3i\\0\end{pmatrix}\right]$$

$$= e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\left[\begin{pmatrix}1\\2\end{pmatrix}+i\begin{pmatrix}-3\\0\end{pmatrix}\right]$$

$$= e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\cos\frac{3}{2}t - \begin{pmatrix}-3\\0\end{pmatrix}\sin\frac{3}{2}t\right] + ie^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\sin\frac{3}{2}t + \begin{pmatrix}-3\\0\end{pmatrix}\cos\frac{3}{2}t\right]$$

$$= e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\cos\frac{3}{2}t - \begin{pmatrix}-3\\0\end{pmatrix}\sin\frac{3}{2}t\right] + ie^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\sin\frac{3}{2}t + \begin{pmatrix}-3\\0\end{pmatrix}\cos\frac{3}{2}t\right]$$

REAL PART: $e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\cos\frac{3}{2}t + \begin{pmatrix}3\\0\end{pmatrix}\sin\frac{3}{2}t\right]$

IMAGINARY PART: $e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\sin\frac{3}{2}t + \begin{pmatrix}-3\\0\end{pmatrix}\cos\frac{3}{2}t\right]$

EACH PART SEPARATELY IS A SOLUTION

Theorem: Suppose $\overrightarrow{\phi(t)} = \overrightarrow{f(t)} + i\overrightarrow{g(t)}$ is a solution to $X' = AX$. Then $\overrightarrow{f(t)}$ and $\overrightarrow{g(t)}$ separately are solutions.

Proof: $\overrightarrow{\phi'(t)} = A\overrightarrow{\phi(t)}$ since $\overrightarrow{\phi(t)}$ is a solution.

Write $\overrightarrow{\phi'} = A\overrightarrow{\phi}$ for short. Thus

$$\overrightarrow{\phi'} - A\overrightarrow{\phi} = \overrightarrow{0}$$

$$[\overrightarrow{f'} + i\overrightarrow{g'}] - A[\overrightarrow{f} + \overrightarrow{ig}] = \overrightarrow{0}$$

$$[\overrightarrow{f'} - A\overrightarrow{f}] + i[\overrightarrow{g'} - A\overrightarrow{g}] = \overrightarrow{0}$$

But a complex number or vector of complex numbers is zero if and only both real and imaginary parts are zero.

Hence

$$\overrightarrow{f'} - A\overrightarrow{f} = \overrightarrow{0} \text{ so } \overrightarrow{f'} = A\overrightarrow{f}$$

and

$$\overrightarrow{g'} - A\overrightarrow{g} = \overrightarrow{0} \text{ so } \overrightarrow{g'} = A\overrightarrow{g}$$

Thus both $\overrightarrow{f}$ and $\overrightarrow{g}$ are solutions.

Procedure for Finding The General Solution of $\mathbf{x}' = A\mathbf{x}$ Where A has Complex Eigenvalues

1. Identify the complex conjugate eigenvalues $\lambda = \mu \pm i\nu$.

2. Determine an eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corresponding to $\lambda = \mu + i\nu$ by solving $(A - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$

3. Express the eigenvector $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{p} + \mathbf{r}i$.

4. Write the solution $\mathbf{x}$ corresponding to $\mathbf{v}$ and separate it into real and imaginary parts:

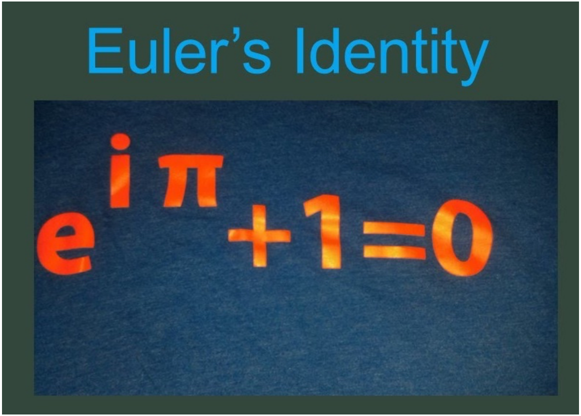
   $$\mathbf{x}(t) = \mathbf{u}(t) + i\,\mathbf{w}(t) \text{ where}$$

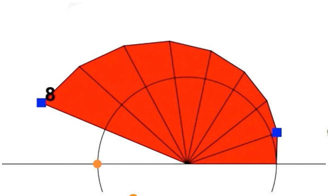
   $$\mathbf{u}(t) = e^{\mu t}(\mathbf{p}\cos\nu t - \mathbf{r}\sin\nu t) \text{ and}$$

   $$\mathbf{w}(t) = e^{\mu t}(\mathbf{r}\cos\nu t + \mathbf{p}\sin\nu t)$$

   Then $\mathbf{u}$ and $\mathbf{w}$ form a linearly independent set of solutions for $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

5. The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = C_1\,\mathbf{u}(t) + C_2\,\mathbf{w}(t)$ where C_1 and C_2 are arbitrary constants.


Euler’s Identity
$e^{i\pi}+1=0$



$$e^{\pi i} = -1$$

Where Did Euler Get The Idea that $e^{i\theta} = \cos\theta + i\sin\theta$?

e^x	$=$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	SO: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$
	$=$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos x$	$=$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$	NOTE $y = \cos x$ IS AN EVEN FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY EVEN POWERS.
	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin x$	$=$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$	NOTE $y = \sin x$ IS AN ODD FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY ODD POWERS.
	$=$	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \dots$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \frac{(ix)^9}{9!} + \frac{(ix)^{10}}{10!} + \frac{(ix)^{11}}{11!} + \frac{(ix)^{12}}{12!} + \dots$$

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \frac{i^8x^8}{8!} + \frac{i^9x^9}{9!} + \frac{i^{10}x^{10}}{10!} + \frac{i^{11}x^{11}}{11!} + \frac{i^{12}x^{12}}{12!} + \dots$$

$$e^{ix} = 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} + \frac{-x^{10}}{10!} + \frac{-ix^{11}}{11!} + \frac{x^{12}}{12!} + \dots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} - \frac{-x^{10}}{10!} - \frac{-ix^{11}}{11!} + \frac{x^{12}}{12!} + \dots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \dots$$

$$e^{ix} =$$
$$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \ldots$$
$$+ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + i\frac{x^9}{9!} - i\frac{x^{11}}{11!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \ldots$$
$$+i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots)$$

$$= \cos x + i \sin x$$