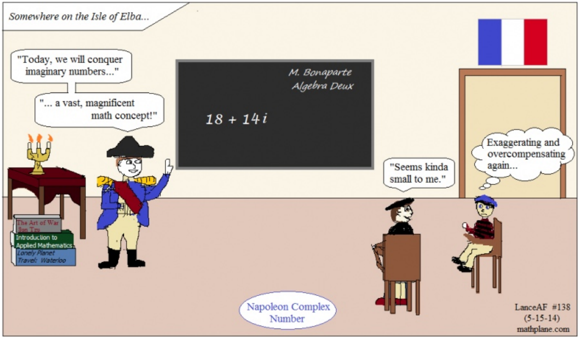
MATH 226: Differential Equations



Class 16: March 28, 2022


WELCOME
BACK

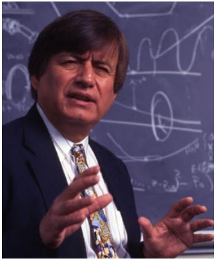
Notes on Assignment 9

Assignment 10

MATLAB Activity Two

Complex Eigenvalues (*Maple* in Handouts folder on CLASSES)

Mathematician of the Week : **Richard Tapia**



March 25, 1938 –

American mathematician and champion of under-represented minorities in the sciences.

Tapia's mathematical research is focused on mathematical optimization and iterative methods for nonlinear problems. His current research is in the area of algorithms for constrained optimization and interior point methods for linear and nonlinear programming.

Current Goal:

Continue Study of Linear Homogeneous Systems With Constant Coefficients

$$X' = A\,X$$

$$2 \times 2 \text{ Case}$$

Theorem: If λ and μ are distinct eigenvalues (real or complex) of a 2×2 matrix A having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x}' = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

Consider the system of first order linear homogeneous differential equations

$$x'(t) = 2x(t) + py(t)$$
$$y'(t) = -1x(t) + 3y(t)$$

where p is any real number.

Then for any initial condition $x(0) = x_0, y(0) = y_0$, there is a unique solution of the system $x = f(t), y = g(t)$ satisfying the initial condition.

The values of $f(t)$ and $g(t)$ will be **real** numbers for all t.

Complex Eigenvalues

Begin with an example **X' = AX** where

$$A = \begin{pmatrix} 2 & p \\ -1 & 3 \end{pmatrix}$$

Here $det(A - \lambda I) = (2 - \lambda)(3 - \lambda) + p = \lambda^2 - 5\lambda + 6 + p$

$$\lambda = \frac{5 \pm \sqrt{25 - 4(6 + p)}}{2} = \frac{5 \pm \sqrt{1 - 4p}}{2}$$

Complex Eigenvalues

$$\lambda = \frac{5 \pm \sqrt{25 - 4(6+p)}}{2} = \frac{5 \pm \sqrt{1-4p}}{2}$$

Some Cases

1. p $= 0$: $\lambda = \frac{5\pm1}{2} = 3$ or 2 (source)
2. p $= 1/4$: $\lambda = \frac{5}{2}$ Double Root (Later This Week)
3. **p = 5/2**: $\lambda = \frac{5\pm\sqrt{1-10}}{2} = \frac{5\pm\sqrt{-9}}{2} = \frac{5\pm3i}{2}$

   $\lambda = \frac{5+3i}{2}$ or $\lambda = \frac{5-3i}{2}$. (**Complex Conjugates**)

   $\lambda = \frac{5}{2} + \frac{3}{2}i$ or $\frac{5}{2} - \frac{3}{2}i$

For a quadratic polynomial, the quadratic formula shows we will have a conjugate pair of roots for $ax^2 + bx + c = 0$ when $b^2 - 4ac < 0$.

Some Basic Facts About Complex Numbers

A **complex number** z is an expression of the form $a+bi$ where a and b are real numbers and $i^2=-1$.

a is called the real part of the complex number,

b is called the imaginary part.

Treat complex numbers as if they were real for the purposes of arithmetic except whenever you encounter ii, replace it with -1.

Arithmetic

Use Associative and Commutative Laws

$$z=a+bi, w=c+di$$

SUM: $z+w=(a+bi)+(c+di)=(a+c)+(b+d)i$

PRODUCT

$$zw=(a+bi)(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i$$

Powers of i

$$i^2 = -1, i^3 = i^2 i = -i, i^4 = i^2 i^2 = (-1)(-1) = 1$$

Thus
$+i = i^1 = i^5 = i^9 = i^{13} = i^{17} = ...$
$-1 = i^2 = i^6 = i^{10} = i^{14} = ...$
$-i = i^3 = i^7 = i^{11} = i^{15} = ...$
$+1 = i^4 = i^8 = i^{12} = i^{16} = ...$

In general, $i^k = i^{k+4}$.

Working with Conjugates

$$\bar{z} = a - bi$$

Then. $\overline{z+w} = \bar{z} + \bar{w}$ (Conjugate of sum is sum of conjugates)

$\overline{zw} = \bar{z}\bar{w}$. (Conjugate of product is product of conjugates)

Note $\overline{z^2} = \overline{zz} = \bar{z}\bar{z} = (\bar{z})^2$.

It follows that if

$$A\vec{v} = \lambda\vec{v}, \text{ then } A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

$\overline{(A\vec{v})} = A\bar{\vec{v}}$ since A is real.

Thus

$$A\bar{\vec{v}} = (\overline{A\vec{v}}) = \overline{\lambda\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

If λ is an eigenvalue of A with eigenvector $\vec{v}$, then $\bar{\lambda}$ is also an eigenvalue of A with eigenvector $\bar{\vec{v}}$

Theorem: If z is a root of a polynomial with real coefficients, then so is $\bar{z}$.

Example: Suppose z is a root of $x^7 - 4x^3 + \pi x - 7$

Then $z^7 - 4z^3 + \pi z - 7 = 0$

Hence $\overline{z^7 - 4z^3 + \pi z - 7} = \bar{0} = 0$

So $\overline{z^7} - \overline{4z^3} + \overline{\pi z} - \bar{7} = 0$

implying $(\bar{z})^7 - 4(\bar{z})^4 - \pi\bar{z} - 7 = 0$

How To Find Eigenvectors

Example:

$$A = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & 3 \end{pmatrix}, \lambda = \frac{5}{2} \pm \frac{3}{2} i.$$

We want $\vec{v}$

$$A - \lambda I = \begin{pmatrix} 2 - \frac{5}{2} - \frac{3}{2} i & \frac{5}{2} \\ -1 & 3 - \frac{5}{2} - \frac{3}{2} i \end{pmatrix} using\, \lambda = \frac{5}{2} + \frac{3}{2} i$$

$$A - \lambda I = \begin{pmatrix} -\frac{1}{2} - \frac{3}{2} i & \frac{5}{2} \\ -1 & \frac{1}{2} - \frac{3}{2} i \end{pmatrix}$$

How To Find Eigenvectors

$$A - \lambda I = \begin{pmatrix} -\frac{1}{2}i - \frac{3}{2}i & \frac{5}{2} \\ -1 & \frac{1}{2} - \frac{3}{2}i \end{pmatrix}$$

First, Check that the determinant is 0:

$$\det(A - \lambda I) = (-\tfrac{1}{2}i - \tfrac{3}{2}i)(\tfrac{1}{2} - \tfrac{3}{2}i) - (-1)(\tfrac{5}{2})$$

$$= -1\tfrac{1}{4} + \tfrac{3}{4}i - \tfrac{3}{4}i - \tfrac{9}{4} + \tfrac{5}{2} = 0.$$

Second, to find a vector $\vec{v}$ with $(A - \lambda I)\vec{v} = \vec{0}$, use the second equation

$$-1v_1 + (\tfrac{1}{2} - \tfrac{3}{2}i)v_2 = 0$$

so $v_1 = \frac{(1-3i}{2} v_2$

Let $v_2 = 2$. Then $v_1 = 1 - 3i$ so $\vec{v} = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix}$

$$\vec{v} = \begin{pmatrix} 1-3i \\ 2 \end{pmatrix}$$

Finally, check that $A\vec{v} = (\frac{5}{2} + \frac{3}{2}i)\vec{v}$:

$$A\vec{v} = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} = \begin{pmatrix} 2-6i+5 \\ -1+3i+6 \end{pmatrix} = \begin{pmatrix} 7-6i \\ 5+3i \end{pmatrix}$$

and

$$\frac{5+3i}{2}\vec{v} = \frac{5+3i}{2} \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5+3i}{2}(1-3i) \\ \frac{5+3i}{2}(2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(5-15i+3i+9) \\ 5+3i \end{pmatrix}$$

$$= \begin{pmatrix} 7-6i \\ 5+3i \end{pmatrix}$$

Apply To System of Differential Equations

$$X' = AX \text{ with } A = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & 3 \end{pmatrix}$$

We have

$$\lambda = \frac{5+3i}{2} \quad \text{so} \quad \mu = \frac{5-3i}{2}$$

$$\vec{v} = \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 1+3i \\ 2 \end{pmatrix}$$

Solutions of Differential Equations Should be

$$e^{\left(\frac{5+3i}{2}\right)t} \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} \text{ and } e^{\left(\frac{5-3i}{2}\right)t} \begin{pmatrix} 1+3i \\ 2 \end{pmatrix}$$

How Can We Make Sense of

$$e^{\left(\frac{5+3i}{2}\right)t} = e^{\left(\frac{5}{2}t + \frac{3i}{2}t\right)}?$$

How Can We Make Sense of

$$e^{\left(\frac{5+3i}{2}\right)t} = e^{\left(\frac{5}{2}t+\frac{3i}{2}t\right)}?$$

By Rules of Exponents, We Should Have

$$e^{\left(\frac{5}{2}t+\frac{3i}{2}t\right)} = e^{\frac{5}{2}t}e^{\frac{3}{2}it}$$

Euler's Formula:

$$e^{bi} = \cos b + i \sin b$$

$$\text{so } e^{\frac{3}{2}it} = \cos\frac{3}{2}t + i\sin\frac{3}{2}t$$

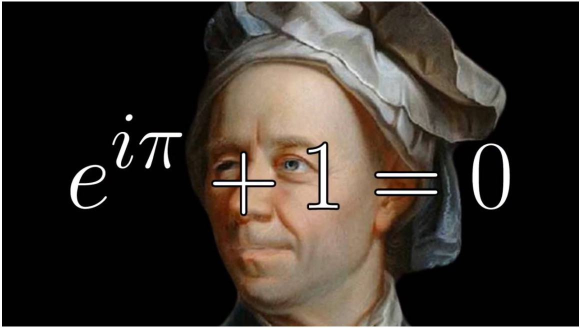
Euler's Formula:

$$e^{bi} = \cos b + i \sin b$$

Note: If $b = \pi$, then

$$e^{\pi i} = \cos \pi + i \sin \pi = -1 + i0 = -1$$

$$\boxed{e^{\pi i} + 1 = 0}$$


$e^{i\pi} + 1 = 0$

Thus

$$e^{\frac{5}{2}t+\frac{3}{2}it}\begin{pmatrix}1-3i\\2\end{pmatrix} = e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\begin{pmatrix}1-3i\\2\end{pmatrix}$$

$$= e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\left[\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-3i\\0\end{pmatrix}\right]$$

$$= e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\left[\begin{pmatrix}1\\2\end{pmatrix} + i\begin{pmatrix}-3\\0\end{pmatrix}\right]$$

$$= e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\cos\frac{3}{2}t - \begin{pmatrix}-3\\0\end{pmatrix}\sin\frac{3}{2}t\right] + ie^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\sin\frac{3}{2}t + \begin{pmatrix}-3\\0\end{pmatrix}\cos\frac{3}{2}t\right]$$

$$= e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\cos\frac{3}{2}t - \begin{pmatrix}-3\\0\end{pmatrix}\sin\frac{3}{2}t\right] + ie^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\sin\frac{3}{2}t + \begin{pmatrix}-3\\0\end{pmatrix}\cos\frac{3}{2}t\right]$$

REAL PART: $e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\cos\frac{3}{2}t + \begin{pmatrix}3\\0\end{pmatrix}\sin\frac{3}{2}t\right]$

IMAGINARY PART: $e^{\frac{5}{2}t}\left[\begin{pmatrix}1\\2\end{pmatrix}\sin\frac{3}{2}t + \begin{pmatrix}-3\\0\end{pmatrix}\cos\frac{3}{2}t\right]$

EACH PART SEPARATELY IS A SOLUTION

Theorem: Suppose $\overrightarrow{\phi(t)} = \overrightarrow{f(t)} + i\overrightarrow{g(t)}$ is a solution to $X' = AX$. Then $\overrightarrow{f(t)}$ and $\overrightarrow{g(t)}$ separately are solutions.

Proof: $\overrightarrow{\phi'(t)} = A\overrightarrow{\phi(t)}$ since $\overrightarrow{\phi(t)}$ is a solution.

Write $\overrightarrow{\phi'} = A\overrightarrow{\phi}$ for short. Thus

$$\overrightarrow{\phi'} - A\overrightarrow{\phi} = \overrightarrow{0}$$

$$[\overrightarrow{f'} + i\overrightarrow{g'}] - A[\overrightarrow{f} + \overrightarrow{ig}] = \overrightarrow{0}$$

$$[\overrightarrow{f'} - A\overrightarrow{f}] + i[\overrightarrow{g'} - A\overrightarrow{g}] = \overrightarrow{0}$$

But a complex number or vector of complex numbers is zero if and only both real and imaginary parts are zero.

Hence

$$\overrightarrow{f'} - A\overrightarrow{f} = \overrightarrow{0} \text{ so } \overrightarrow{f'} = A\overrightarrow{f}$$

and

$$\overrightarrow{g'} - A\overrightarrow{g} = \overrightarrow{0} \text{ so } \overrightarrow{g'} = A\overrightarrow{g}$$

Thus both $\overrightarrow{f}$ and $\overrightarrow{g}$ are solutions.

Procedure for Finding The General Solution of $\mathbf{x}' = A\mathbf{x}$ Where A has Complex Eigenvalues

1. Identify the complex conjugate eigenvalues $\lambda = \mu \pm i\nu$.

2. Determine an eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corresponding to $\lambda = \mu + i\nu$ by solving $(\boldsymbol{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$

3. Express the eigenvector $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{p} + \mathbf{r}i$.

4. Write the solution $\mathbf{x}$ corresponding to $\mathbf{v}$ and separate it into real and imaginary parts:

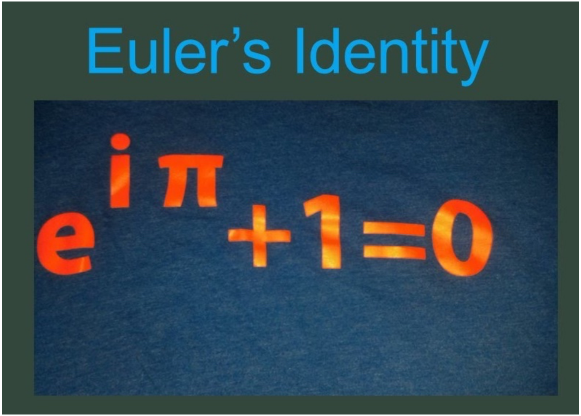
$$\mathbf{x}(t) = \mathbf{u}(t) + i\,\mathbf{w}(t) \text{ where}$$

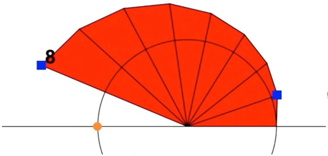
$$\mathbf{u}(t) = e^{\mu t}(\mathbf{p}\cos\nu t - \mathbf{r}\sin\nu t) \text{ and}$$

$$\mathbf{w}(t) = e^{\mu t}(\mathbf{r}\cos\nu t + \mathbf{p}\sin\nu t)$$

   Then $\mathbf{u}$ and $\mathbf{w}$ form a linearly independent set of solutions for $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

5. The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = C_1\,\mathbf{u}(t) + C_2\,\mathbf{w}(t)$ where C_1 and C_2 are arbitrary constants.


Euler's Identity
$e^{i\pi}+1=0$



$$e^{\pi i} = -1$$

Where Did Euler Get The Idea that $e^{i\theta} = \cos\theta + i\sin\theta$?

e^x	$=$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	SO: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$
	$=$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos x$	$=$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$	NOTE $y = \cos x$ IS AN EVEN FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY EVEN POWERS.
	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin x$	$=$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$	NOTE $y = \sin x$ IS AN ODD FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY ODD POWERS.
	$=$	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \ldots$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \frac{(ix)^9}{9!} + \frac{(ix)^{10}}{10!} + \frac{(ix)^{11}}{11!} + \frac{(ix)^{12}}{12!} + \ldots$$

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \frac{i^8x^8}{8!} + \frac{i^9x^9}{9!} + \frac{i^{10}x^{10}}{10!} + \frac{i^{11}x^{11}}{11!} + \frac{i^{12}x^{12}}{12!} + \ldots$$

$$e^{ix} = 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} + \frac{-x^{10}}{10!} + \frac{-ix^{11}}{11!} + \frac{x^{12}}{12!} + \ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} - \frac{-x^{10}}{10!} - \frac{-ix^{11}}{11!} + \frac{x^{12}}{12!} + \ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \ldots$$

$$e^{ix} =$$
$$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \ldots$$
$$+ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + i\frac{x^9}{9!} - i\frac{x^{11}}{11!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \ldots$$
$$+i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots)$$

$$= \cos x + i \sin x$$