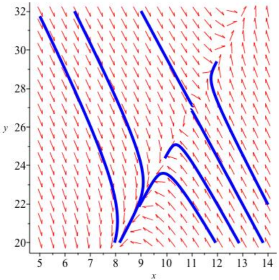
MATH 226: Differential Equations



Class 14: March 16, 2022

Notes on Exam 1

Existence and Uniqueness Theorems for Linear **Systems**

Doomsday Model Data

Complex Numbers

(Also See Course Website)

Announcements

Mathematician of the Day

Lewis F. Richardson

October 11, 1881– September 30, 1953

The equations are merely a description of what people would do if they did not stop and think.

Today's Topics

More Analysis of The Richardson Arms Race Model

More About Systems of Two First Order Linear Differential Equations With Constant Coefficients

Richardson Arms Race Model

Lewis F. Richardson

Arms And Insecurity:

A Mathematical Study Of The Causes And Origins Of War

$x(t)$ = **Arms Expenditure of Blue Nation**

$y(t)$ = **Arms Expenditure of Red Nation**

$$x' = ay - mx + r$$

$$y' = bx - ny + s$$

where a, b, m, n are positive constants while r and s are constants.

Structure: $\vec{X} = A\vec{X} + \vec{b}$ or $\mathbf{x'} = A\mathbf{x} + \mathbf{b}$

$$\mathbf{X'} = A\,\mathbf{X} + \mathbf{b}$$

Make Change of Variables

$$X = x - x^*$$

$$Y = y - y^*$$

where $ay^* - mx^* + r = 0, bx^* - ny^* + s = 0$

To Convert To Homogeneous System of Form

$$\mathbf{X'} = A\,\mathbf{X}$$

$$\mathbf{X'} = A\,\mathbf{X}$$

where

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix} \text{ has solution}$$

$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w}$$

where α and β are arbitrary constants
λ is an eigenvalue of A with associated eigenvector $\vec{v}$ and
$\mu \neq \lambda$ is an eigenvalue of A with associated eigenvector $\vec{w}$.

The solution of the original system is then

$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w} + \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

Two Particular Examples:

$x' = -5x + 4y + 1$ $y' = 3x - 4y + 2$	$x' = 11y - 9x - 15$ $y' = 12x - 8y - 60$
$(x^*, y^*) = (\frac{3}{2}, \frac{13}{8})$	$(x^*, y^*) = (13, 12)$
$A = \begin{bmatrix} -5 & 4 \\ 3 & -4 \end{bmatrix}$ $\lambda = -1, \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mu = -8, \vec{w} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$	$A = \begin{bmatrix} -9 & 11 \\ 12 & -8 \end{bmatrix}$ $\lambda = 3, \vec{v} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$ $\mu = -20, \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\alpha e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{-8t} \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ \frac{13}{8} \end{bmatrix}$	$\alpha e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + \beta e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 13 \\ 12 \end{bmatrix}$

Existence and Uniqueness Theorems for Linear Systems

Theorem 2.4.1: If $p(t)$ and $g(t)$ are continuous functions on an open interval I containing the point $t = t_o$ and y_o is any prescribed initial value, then there exists a unique solution $y = \phi(t)$ of the differential equation that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for all t in I with $\phi(t_o) = y_o$.

Theorem 3.2.1: If $P(t)$ is an $n \times n$ matrix and $\mathbf{g}(t)$ is an $n \times 1$ vector whose entries are continuous on an open interval I containing the point t_o and $\mathbf{y_o}$ is any prescribed initial value, then there is a unique solution $\mathbf{y} = \mathbf{\Phi}(t)$ of the system of differential equations

$$\boldsymbol{X}' = P(t)\boldsymbol{X} + \mathbf{g}(t)$$

for all t in I with $\mathbf{\Phi}(t_o) = \mathbf{y_o}$.

Theorem 3.2.1: If $P(t)$ is an $n \times n$ matrix and $\mathbf{g}(t)$ is an $n \times 1$ vector whose entries are continuous on an open interval I containing the point t_o and $\mathbf{y_0}$ is any prescribed initial value, then there is a unique solution $\mathbf{y} = \mathbf{\Phi}(t)$ of the system of differential equations

$$\boldsymbol{X}' = P(t)\boldsymbol{X} + \mathbf{g}(t)$$

for all t in I with $\mathbf{\Phi}(t_o) = \mathbf{y_0}$.

What Does This Theorem Say in the case $P(t)$ is an $n \times n$ matrix of **constants** and $\mathbf{g(t)}$ is identically 0?

There is a unique solution valid for all real numbers!

Focus on Linear Homogeneous System with Constant Coefficients

$$\mathbf{X'} = A\,\mathbf{X}$$

where A is a 2×2 matrix.

Begin with Earlier Example

$$x' = -9x + 11y$$
$$y' = 12x - 8y$$

$$A = \begin{bmatrix} -9 & 11 \\ 12 & -8 \end{bmatrix}$$

Two solutions to the homogeneous system are

$$e^{3t}\vec{v} \text{ and } e^{-20t}\vec{w}$$

$$e^{3t}\begin{bmatrix}11\\12\end{bmatrix} \text{ and } e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$$

Then $C_1e^{3t}\begin{bmatrix}11\\12\end{bmatrix} + C_2e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$ is a solution for any constants C_1 and C_2.

Now suppose $\Phi(t)$ is any solution to the system with $\Phi(0) = \begin{bmatrix}x_0\\y_0\end{bmatrix}$

CLAIM: We can find $C1$ and $C2$ so that

$$\Phi(t) = C_1e^{3t}\begin{bmatrix}11\\12\end{bmatrix} + C_2e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$$

CLAIM: We can find $C1$ and $C2$ so that

$$\Phi(t) = C_1 e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

NEED: Agreement at $t = 0$:

$$C_1 e^{3\times 0} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 e^{-20\times 0} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$C_1 \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$11C_1 + 12C2 = x_0$$
$$12C_1 - 1C_2 = y_0$$

$$\begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

has a solution for all x_0, y_0 exactly when the coefficient matrix $M = \begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix}$ is invertible

and this happens if and only the columns of the coefficient matrix are a linearly independent set of vectors.

But the columns are $\vec{v}$ and $\vec{w}$ which are eigenvectors belonging to distinct eigenvalues

so they do form a linearly independent set.

The solution will be

$$\begin{bmatrix} C1 \\ C2 \end{bmatrix} = M^{-1} \begin{bmatrix} x_0 \\ y_o \end{bmatrix} = \frac{-1}{23} \begin{bmatrix} -1 & -1 \\ -12 & 11 \end{bmatrix} \begin{bmatrix} x_0 \\ y_o \end{bmatrix}$$

The solution will be

$$\begin{bmatrix} C1 \\ C2 \end{bmatrix} = M^{-1} \begin{bmatrix} x_0 \\ y_o \end{bmatrix} = \frac{-1}{23} \begin{bmatrix} -1 & -1 \\ -12 & 11 \end{bmatrix} \begin{bmatrix} x_0 \\ y_o \end{bmatrix}$$

$$\text{Thus } \begin{bmatrix} C1 \\ C2 \end{bmatrix} = \begin{bmatrix} \frac{x_0+y_0}{23} \\ \frac{12x_0-11y_0}{23} \end{bmatrix}.$$

We have found one solution of the homogeneous system that

- Agrees with Φ and $t = 0$ and
- Is a linear combination of $e^{3t}\vec{v}$ and $e^{-20t}\vec{w}$.

By The Uniqueness Theorem, Φ **must** be a linear combination of these two solutions.

Thus these two particular solutions are a **Spanning Set** for the collection of all solutions to the homogeneous system.

The two particular solutions $e^{3t}\vec{v}$ and $e^{-20t}\vec{w}$ form a **Spanning Set** for the collection of all solutions to the homogeneous system.

What Made This Work?

$\vec{v}, \vec{w}$ is a linearly independent set of vectors
which we know is true since they are associated with two distinct eigenvalues.

Moreover, the two solutions themselves are Linearly Independent Solutions. They form a **BASIS** for the set of all solutions to the homogeneous system of differential equations **X'** $=A$ **X**.

Theorem: Let λ and μ be distinct eigenvalues for a square matrix A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$. Then $e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}$ is a linearly independent set of solutions for **X'** $=A$ **X**.

Theorem: Let λ and μ be distinct eigenvalues for a square matrix A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$.

Then $\{e^{3t}\vec{v},\ e^{-20t}\vec{w}\}$ is a Linearly Independent set of solutions of $\mathbf{X'} = A\,\mathbf{X}$.

Proof: Suppose there are constants C_1 and C_2 such that

$$C_1 e^{\lambda t}\vec{v} + C_2 e^{\mu t}\vec{w} = \vec{0}$$

for all t where $\vec{0}$ is the function identically equal to the zero vector for all t.

Evaluate this identity at $t = 0$:

$$C_1\vec{v} + C_2\vec{w} = 0$$

BUT $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors.

Hence it must be that $C_1 = 0$ and $C_2 = 0$

We used the fact that $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors to prove that

- $\{e^{3t}\vec{v},\ e^{-20t}\vec{w}\}$ is a spanning set for the solutions of $\mathbf{X'} = A\,\mathbf{X}$ and
- $\{e^{3t}\vec{v},\ e^{-20t}\vec{w}\}$ is a linearly independent set of solutions of $\mathbf{X'} = A\,\mathbf{X}$

The Linear Independence of $\{\vec{v}, \vec{w}\}$ followed from the fact that they were associated with distinct (unequal) eigenvalues.

WE CAN DO THE SAME THING FOR ANY SUCH SYSTEM WHERE THE MATRIX HAS TWO DISTINCT EIGENVALUES

Another Linear Homogeneous System with Constant Coefficients

X' $= A$ **X** where A is a 2×2 matrix.

$$x' = 2x + 1y$$
$$y' = -3x + 6y$$

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}$$

Characteristic Equation $(\det A - \lambda I) = 0$ is

$$\lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5) = 0$$

so eigenvalues are $\lambda = 3, \mu = 5$

and solution to the systems of first order differential equations is

$$C_1 e^{3t}\vec{v} + C_2 e^{5t}\vec{w}$$

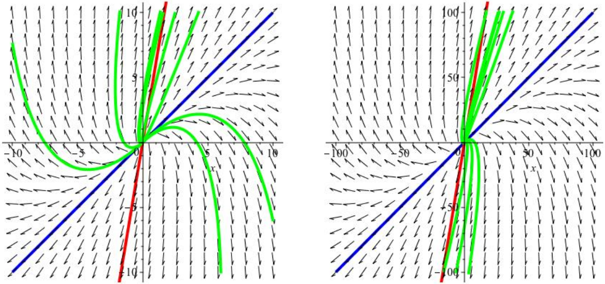
where C_1, C_2 are arbitrary constants and

$\vec{v}, \vec{w}$ are eigenvectors associated with $\lambda = 3$ and $\lambda = 5$, respectively.

We can write the general solution
$C_1 e^{3t}\vec{v} + C_2 e^{5t}\vec{w}$ as $e^{5t}(C_1 e^{-2t}\vec{v} + C_2\vec{w})$

If $C_2 = 0$, then solution does what as t gets large?
Moves along the vector $\vec{v}$.

If $C_2 \neq 0$, then what does the solution do as t gets large?
Approaches the vector $\vec{w}$.



$$x' = 2x + 1y, \; y' = -3x + 6y$$

$$C_1 e^{3t} \vec{v} + C_2 e^{5t} \vec{w} = e^{5t} (C_1 e^{-2t} \vec{v} + C_2 \vec{w})$$

Median: 91

Average: 88.8