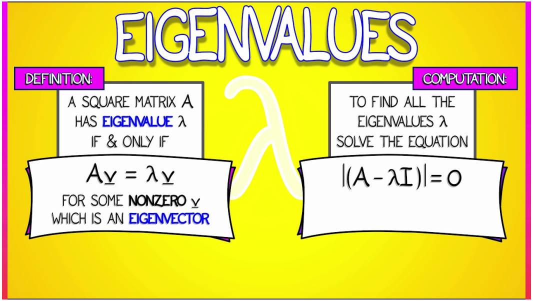
# MATH 226 Differential Equations




March 11, 2022

# Assignment 8
# Eigenvalues in MATLAB

# Announcements

- First Team Projects Due Today ? 5 PM
- Exam 1
  - Monday
  - 7 PM - ? (No Time Limit)
  - 229 Axinn
  - No Calculators, Books, Notes, Smart Phones, etc.
  - Focus on Material in Chapters 1 and 2

## Today's Topics

### Introduction To Systems of First Order Differential Equations

- Lotka – Volterra Predator Prey Model
- Richardson Arms Race Model
- Kermack – McKendrick Epidemic Model
- Home Heating Model
- Terrorism Recruitment Model

Initial Concern: **Homogeneous Systems of 2 First Order Differential Equations With Constant Coefficients**

$$x' = ax + by,\ y' = cx + dy$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{X'} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{X}$$

**Ideas and Tools From Linear Algebra Are Essential To This Study**

## EIGENVALUES AND EIGENVECTORS

$A \quad n \times n \quad$ [Square Matrix ]

$\vec{x} \quad n \times 1 \quad$ [Element of $R^n$ ]

Then $A\vec{x}$ is another vector in $R^n$.

Is there a **nonzero** vector $\vec{v}$ and a constant $\lambda$ such that

$$A\vec{v} = \lambda\vec{v}?$$

The equation $A\vec{v} = \lambda\vec{v}$ is equivalent to

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

or $(A - \lambda I)\vec{v} = \vec{0}$

which is a system of homogeneous equations.

**The system has nontrivial solution if and only if $(A - \lambda I)$ is Non-Invertible.**

$$\Rightarrow det(A - \lambda I) = 0.$$

Example

$$x' = -13x + 6y$$
$$y' = 2x - 2y$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\vec{x}' = A\vec{x}$ or **X'** = A **X**

Looks like $x' = ax$ which has solution $x = Ce^{at}$.

Could there be a scalar $\lambda$ and **nonzero** vector $\vec{v}$ such that $\vec{x} = e^{\lambda t}\vec{v}$ is a solution?

$\vec{x}' = A\vec{x}$ becomes $\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v}$

## Finding Eigenvalues and Associated Eigenvectors

$$\text{Example}: \; A = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -13-\lambda & 6 \\ 2 & -2-\lambda \end{bmatrix}$$

$$\det\,(A - \lambda I) = (-13-\lambda)(-2-\lambda) - (2)(6)$$

$$\det\,(A - \lambda I) = 26 + 13\lambda + 2\lambda + \lambda^2 - 12$$

$$\det\,(A - \lambda I) = \lambda^2 + 15\lambda + 14 = (\lambda + 14)(\lambda + 1)$$

$$\lambda = -14 \text{ or } \lambda = -1.$$

$$A = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix}$$

For $\lambda = -1$, $A - \lambda I = \begin{bmatrix} -13 - (-1) & 6 \\ 2 & -2 - (-1) \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -12 & 6 \\ 2 & -1 \end{bmatrix}$$

Row Reduces to $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$

$-2v_1 + v_2 = 0$ so $v_2 = 2v_1$

so a corresponding eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

One Solution to $\vec{x}' = A\vec{x}$ is $e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} -13 & 6 \\ 2 & -2 \end{bmatrix}$$

For $\lambda = -14$, $A - \lambda I = \begin{bmatrix} -13 - (-14) & 6 \\ 2 & -2 - (-14) \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 6 \\ 2 & 12 \end{bmatrix}$$

Row Reduces to $\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix}$

$$w_1 + 6w_2 = 0 \text{ so } w_2 = -\frac{1}{6} w_1$$

so a corresponding eigenvector is $\begin{bmatrix} 6 \\ -1 \end{bmatrix}$.

Another Solution to $\vec{x}' = A\vec{x}$ is $e^{-14t} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

Suppose $X_1$ and $X_2$ are each solutions of $\mathbf{X'} = \mathrm{A}\mathbf{X}$ .
Let $\alpha$ and $\beta$ be any two constants.
Claim $\alpha X_1 + \beta X_2$ is also a solution

Proof: On One Hand:
$$(\alpha X_1 + \beta X_2)' = \alpha X_1' + \beta X_2' = \alpha A X_1 + \beta A X_2$$
On Other Hand :
$$A(\alpha X_1 + \beta X_2) = \alpha A X_1 + \beta A X_2$$

**The set of solutions to $\mathbf{X'} = \mathrm{A}\mathbf{X}$ is a VECTOR SPACE.**

**Theorem**: Suppose $\lambda \neq \mu$ are two distinct eigenvalues of a square matrix $A$ with respective eigenvectors $\vec{v}$ and $\vec{w}$; That is,

$$A\vec{v} = \lambda\vec{v} \text{ and } A\vec{w} = \mu\vec{w}$$

Then $\{\vec{v}, \vec{w}\}$ is a Linearly Independent set of vectors.

Proof: Suppose $a$ and $b$ are constants such that

$$(*)\ a\vec{v} + b\vec{w} = \vec{0}$$

First, Multiply (*) by $A$:

$$aA\vec{v} + bA\vec{w} = A\vec{0} = \vec{0}$$

$$(**)\ a\lambda\vec{v} + b\mu\vec{w} = \vec{0}$$

Next, Multiply (*) by $\mu$ to obtain

$$(***)\ a\mu\vec{v} + b\mu\vec{w} = \vec{0}$$

Now subtract (***) from (**):

$$a(\lambda - \mu)\vec{v} = \vec{0}$$

Since $\lambda \neq \mu$ and $\vec{v} \neq \vec{0}$, we must have $a = 0$.
But this means $b\vec{w} = \vec{0}$ and hence $b = 0$.