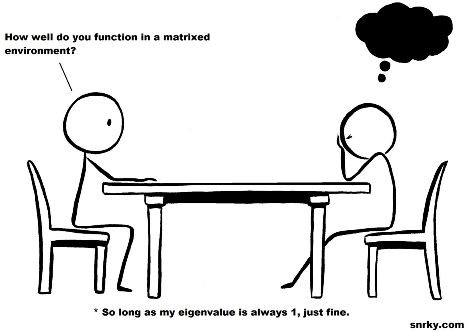
MATH 226 Differential Equations



March 9, 2022

Notes on Assignment 6

Notes on Sample Exam 1

Project 1 Peer and Self Evaluations

Announcements

- First Team Projects Due Friday
- Exam 1
  - Monday, March 14
  - 7 PM - ? (No Time Limit)
  - No Calculators, Books, Notes, Smart Phones, etc.
  - Focus on Material in Chapters 1 and 2

Mathematicians of the Day

Norman Routledge

March 7, 1928 – April 27, 2013

Today's Topics

Introduction To Systems of First Order Differential Equations

Lotka – Volterra Predator Prey Model

$$x(t) = \text{Population of Prey}$$
$$y(t) = \text{Population of Predator}$$

$$x' = ax - bxy$$

$$y' = mxy - ny$$

where a, b, m, n are positive constants.

Richardson Arms Race Model

$x(t) =$ Arms Expenditure of Blue Nation
$y(t) =$ Arms Expenditure of Red Nation

$$x' = ay - mx + r$$

$$y' = bx - ny + s$$

where a, b, m, n are positive constants while r and s are constants.

Kermack – McKendrick Epidemic Model

(1927 – 1939)

$$\boxed{\text{Susceptibles}} \rightarrow \boxed{\text{Infectives}} \rightarrow \boxed{\text{Removeds}}$$

S = Number of Susceptibles
I = Number of Infectives
R = Number of Recovereds

$$S' = -\beta SI$$

$$I' = \beta SI - rI$$

$$R' = rI$$

Home Heating Model

$x(t) =$ Temperature of the Ground Floor
$y(t) =$ Temperature of Upper Floor
$k =$ coefficient of thermal conductivity
$k_1 =$ Thermal Conductivity of Floor on Ground Level
$k_2 =$ Thermal Conductivity of Ceiling on Ground Level
$k_3 =$ Thermal Conductivity of Walls on Ground Level
$k_4 =$ Thermal Conductivity of Walls on Upper Floor and Roof
$T_e =$ Temperature of Surrounding Environment

$$x' = -(k_1 + k_2 + k_3)x + k_2 y + k_1 T_g + k_3 T_e(t) + f(t)$$
$$y' = k_2 x - (k_2 + k_4)y + k_4 T_e(t)$$

where $f(t)$ describes the heat source located on the ground floor.

Terrorism

x = Number of Terrorists at time t
y = Number of Individuals Who Can Be Influenced By Terrorist Propaganda and Counter – Terrorist Influence.
z = Number of Individuals Resistant To Terrorist Propaganda

$$x' = ay - bx^2 + (c - 1)$$

$$y' = -axy - cx^2y + fx + gy$$

$$z' = ex^2y - hx + mz$$

Our Focus: Linear Models

$$x' = a(t)x + b(t)y + f(t), \; y' = c(t)x + d(t)y + g(t)$$

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Narrow Focus First to **Homogeneous Systems**

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Initial Concern: **Homogeneous Systems of 2 First Order**

Differential Equations With Constant Coefficients

$$x' = ax + by,\ y' = cx + dy$$

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Differential Equations With Constant Coefficients

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has form **X'** = A **X** where **X** $= \begin{bmatrix} x(t) \\ y(y) \end{bmatrix}$

Ideas and Tools From Linear Algebra Are Essential To This Study

Initial Concern: **Homogeneous Systems of 2 First Order Differential Equations With Constant Coefficients**

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Start With System of Linear **ALGEBRAIC** Equations

Two Linear Equations in 2 Unknowns

$$3x + 4y = 18$$

$$5x - 2y = 4$$

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1 Solution　　No Solutions　　Infinitely Many Solutions

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Matrix Representation

$$\begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \end{bmatrix}$$

$$A \quad \mathbf{X} = \mathbf{b}$$

Reduce A to Row Echelon Form

Example

$$3x_1 + 5x_2 - 4x_3 = 0$$

$$-3x_1 - 2x_2 + 4x_3 = 0$$

$$6x_1 + x_2 - 8x_3 = 0$$

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One Dimensional Set of Solutions

EIGENVALUES AND EIGENVECTORS

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$A \quad n \times n \quad$ [Square Matrix]

$\vec{x} \quad n \times 1 \quad$ [Element of R^n]

Then $A\vec{x}$ is another vector in R^n.

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$$\Rightarrow det(A - \lambda I) = 0.$$