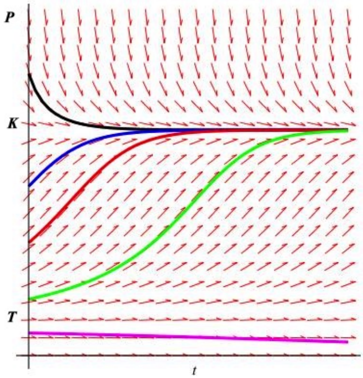
MATH 226 Differential Equations



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Qualitative Analysis of Single Species Population Dynamics
Autonomous Models
y' = f(y) or P' = f(P)

Logistic Growth With Threshold

$$P' = rP(1 - \frac{P}{K})(\frac{P}{T} - 1), 0 < T < K$$

K is Carrying Capacity and T is Threshold.

Development and Qualitative Analysis of Logistic Model of Population Growth With a Threshold

Model I: Simple Exponential Growth

$$P' = rP, r > 0$$

The parameter r is called the *Intrinsic Growth Rate*

Model assets that population grows at a constant percentage rate.

Equivalent to *Constant Per Capita Growth:*

$$\frac{P'}{P} = r$$

Predicts: Population will grow more and more rapidly over time and be an **unbounded** function of t.

Model is often realistic in early stages of growth, but fails in long term as we have only a finite amount of resources.

Solution is

$$P(t) = P(0)e^{rt}$$

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Model II: Logistic Growth

Population growth must slow down as population becomes large. There is a *Carrying Capacity*, the largest population the environment can sustain.

$$P' = rP\left(1 - \frac{P}{K}\right), r, K > 0$$

When P is small compared to K, $1 - \ fracPK$ is close to 1 so we have near exponential growth for small populations. As P approaches K, P' gets closer to 0.

P' is positive for $0 < P < k$ and negative if $P > K$.

See our text for more analysis.

Model III: Logistic Growth with a Threshold

If the population is very small, then it may difficult to find a mating partner and population will actually decrease; perhaps even move toward extinction

$$P' = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{T} - 1\right), 0 < T < K$$

If P is smaller than T, then $\frac{P}{T} - 1$ will be negative and P will decrease.

T is called a *threshold* and typically is at least a couple of orders of magnitude smaller than K

Think of $K = 1000$, $T = 10$ for an example.

When is P Increasing and When is it Decreasing?

Analyze Sign of P'. Since P is nonnegative, the sign of P' is the same as the sign of the product $\left(1 - \frac{P}{K}\right)\left(\left(\frac{P}{T} - 1\right)\right)$.

The graph of this product is a downward opening parabola so it is positive only when $T < P < K$.

Thus we have

$0 < P < T$	$P' < 0$	P decreasing
$T < P < K$	$P' > 0$	P increasing
$P > K$	$P' < 0$	P decreasing

To Determine Concavity, We Need To Examine Sign of P''
Idea: Multiply out expression for P' to get a polynomial.

$$P' = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{T} - 1\right) = r\left(-\frac{P^3}{KT} + P^2\left(\frac{T+K}{KT}\right) - P\right)$$

Now differentiate term by term, using the Chain Rule:

$$P'' = r\left(\frac{-3P^2}{KT} + 2\frac{T+K}{KT}P - 1\right)P'$$

We know the sign of r $(+)$ and the sign of P' for all values of P.
We need to find the sign of

$$Q(P) = \frac{-3P^2}{KT} + 2\frac{T+K}{KT}P - 1$$

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The graph of Q as a function of P is a downward opening parabola. It has two roots $a < b$ so $Q(P)$ is positive for $a < P < b$ and negative for other values of P.

To fins a and b, we need to solve $Q(P) = 0$

$$\frac{-3P^2}{KT} + 2\frac{T+K}{KT}P - 1 = 0$$

Multiply through by KT to clear fractions:

$$-3P^2 + 2(T+K)P - KT = 0$$

Apply Quadratic Formula to

$$-3P^2 + 2(T+K)P - KT = 0$$

$$P = \frac{-2(T+K) \pm \sqrt{4(T+K)^2 - 4(3)KT}}{-6}$$

which simplifies to

$$P = \frac{(T+K) \pm \sqrt{(T+K)^2 - 3KT}}{3}.$$

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Root b: Take Positive Square Root: First observe that $3KT$ is relatively very small compared to $(T+K)^2$

(For $T = 10, K = 1000$, we have $3KT = 30,000$ while $T + K)^2$ is well above 1,000,000)

Thus $(T+K)^2 - 3KR \approx (T+K)^2$ so

$\sqrt{(T+K)^2 - 3KT} \approx (T+K)$

and $b \approx \frac{(T+K)+(K+T)}{3} = \frac{2}{3}(K+T) \approx \frac{2}{3}K$

since K is much larger than T.

(For $T = 10, K = 1000$, exact solution is 668.35)

Root a: Take Negative Square Root:

$$a = \frac{(T+K) - \sqrt{(T+K)^2 - 3KT}}{3}$$

We will be subtracting from $T + K$ a number slightly smaller than $T + K$ so we expect the answer to be quite small.
Using the tangent line approximation $f(x+h) \approx f(x) + f'(x)h$ with the square root function $f(x) = \sqrt{x}$ we have

$$\sqrt{x+h} \approx \sqrt{x} + \frac{h}{2\sqrt{x}}.$$

Letting $x = (T+K)^2$ and $h = -3KT$, we have

$$\sqrt{(T+K)^2 - 3KT} \approx (T+K) - \frac{3KT}{2(T+K)}$$

Thus $a \approx \dfrac{(T+K) - (T+K) + \frac{3KT}{2(T+K)}}{3} = \dfrac{KT}{2(T+K)}$

We have $a \approx \frac{KT}{2(T+K)}$ but $T + K \approx K$ so

$$a \approx \frac{KT}{2K} = \frac{T}{2}$$

(For $T = 10, K = 1000$, exact solution is 4.99)
Thus the roots of $Q(P)$ are approximately $\frac{1}{2}T$ and $\frac{2}{3}K$.
So $Q(P)$ is positive for P roughly between $\frac{1}{2}T$ and $\frac{2}{3}K$ and negative elsewhere,

P	$Q(P)$	P'	P''
$0 < P < \frac{T}{2}$	Negative	Negative	Positive
$\frac{T}{2} < P < T$	Positive	Negative	Negative
$T < P < \frac{2}{3}K$	Positive	Positive	Positive
$\frac{2}{3}K < P < K$	Negative	Positive	Negative
$P > K$	Negative	Negative	Positive

Summarizing Our Results:

P	Graph of P
$0 < P < \frac{T}{2}$	Decreasing, Concave Up
$\frac{T}{2} < P < T$	Decreasing, Concave Down
$T < P < \frac{2}{3}K$	Increasing, Concave Up
$\frac{2}{3}K < P < K$	Increasing, Concave Down
$P > K$	Decreasing, Concave Up

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$\frac{2}{3}K < P < K$	Increasing, Concave Down
$P > K$	Decreasing, Concave Up