A Unified Treatment of Tangent Lines and Tangent Planes

Tangent Lines to Curves

Case I: $f: \mathbb{R}^1 \to \mathbb{R}^1$ (Coordinate representation of the curve)

Here the **graph** of f is the curve and y = f(x) is equation of the curve The tangent line to the graph of f at x = a is tangent line at (a, f(a))and has equation y = f(a) + f'(a)(x - a)

Case 2: $f: \mathbb{R}^1 \to \mathbb{R}^2$ (Parametric Representation of the curve)

Here the *image* of *f* is the curve

We focus on the tangent line to the image of f.

Here
$$\mathbf{f}(t) = (f_1(t), f_2(t))$$
 and $\mathbf{f}'(t) = (f_1'(t), f_2'(t))$

Then: Tangent line is set of vectors of the form $\mathbf{f}(a) + \mathbf{f}'(a)t$

Case 3: $f: \mathbb{R}^1 \to \mathbb{R}^m$

Here the *image* of *f* is the curve

Now
$$\mathbf{f}(t) = (f_1(t), f_2(t), ..., f_m(t))$$
 and $\mathbf{f}'(t) = (f_1'(t), f_2'(t), ..., f_m'(t))$

Again, Tangent line is set of vectors of the form $\mathbf{f}(a) + \mathbf{f}'(a)t$

Tangent Planes to Surfaces

Case 1: $f: \mathbb{R}^2 \to \mathbb{R}^1$ (coordinate representation of the curve)

Here the *graph* of *f* is the surface

Now z = f(x, y) is the equation of the surface.

If (a,b) is a point in \mathbb{R}^2 in the domain of f, then the tangent plane to the surface at (a,b,f(a,b)) is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Note that we can write this equation in a number of equivalent ways:

$$z = f(a,b) + \left(f_x(a,b), f_y(a,b)\right) \bullet (x-a, y-b)$$

$$z = f(a,b) + \left(f_x(a,b), f_y(a,b)\right) \bullet \left((x,y) - (a,b)\right)$$

$$z = f(\mathbf{a}) + f'(\mathbf{a}) \bullet (\mathbf{x} - \mathbf{a})$$
 where $\mathbf{a} = (a, b), \mathbf{x} = (x, y),$ and $f' = (f_x, f_y) = \nabla f$

Case 2: $f: \mathbb{R}^2 \to \mathbb{R}^3$ (parametric representation of the surface)

Here the **image** of *f* is the surface

Now
$$f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$$

In vector form, with $\mathbf{a} = (u, v)$, and $f(\mathbf{a}) = (f_1(\mathbf{a}), f_2(\mathbf{a}), f_3(\mathbf{a}))$

Let
$$\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{pmatrix}$$
, $\frac{\partial f}{\partial v} = \begin{pmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial v} \end{pmatrix}$, and $f' = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}$

The tangent plane to surface at $f(\mathbf{a})$ is the set of vectors of the form $f(\mathbf{a}) + f'(\mathbf{a}) \begin{pmatrix} u \\ 1 \end{pmatrix}$

$$\operatorname{Or} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{a}) + \frac{\partial f_1}{\partial u}(\mathbf{a})u + \frac{\partial f_1}{\partial v}(\mathbf{a})v \\ f_2(\mathbf{a}) + \frac{\partial f_2}{\partial u}(\mathbf{a})u + \frac{\partial f_2}{\partial v}(\mathbf{a})v \\ f_3(\mathbf{a}) + \frac{\partial f_3}{\partial u}(\mathbf{a})u + \frac{\partial f_3}{\partial v}(\mathbf{a})v \end{pmatrix} \quad \text{We may solve the first two equations} \quad \text{for } u \text{ and } v \text{ in terms of } x \text{ and } y. \quad \text{Then substitute into third equation}$$

o get z in terms of x and y.

Summary: Equations for the Tangent Objects

	Curve	Surface
Coordinate Form	y = f(a) + f'(a)(x - a)	$z = f(\mathbf{a}) + f'(\mathbf{a}) \bullet (\mathbf{x} - \mathbf{a})$
Parametric Form	$\mathbf{f}(a) + \mathbf{f}'(a)t$	$f(\mathbf{a}) + f'(\mathbf{a}) \begin{pmatrix} u \\ v \end{pmatrix}$
	Tangent is a Line	Tangent is a Plane