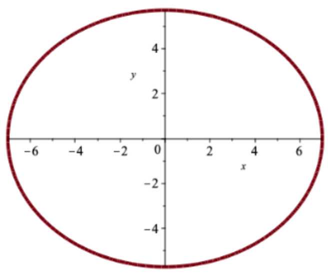
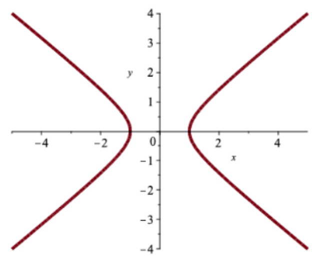
MATH 223 Spring 2022

Some Notes on Assignment 16

1. Let $f(x, y) = 2x^2 + 3y^2$ be a real-valued function defined on the plane.
   (a) Let **P** be the point (5,4) and *C* the level curve of *f* containing **P**. Identify the nature of *C:* Is it a Circle? Parabola? Pair of Lines? Sketch a picture of *C.*
   Solution: $f(5,4) = 2(5^2) + 3(4^2) = 50 + 48 = 98.$ $2x^2 + 3y^2$ =98 is the equation of an ellipse. Below left is a graph of the level curve.



Level Curve For Exercise 1



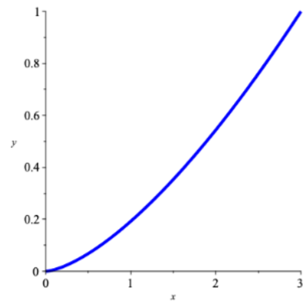
Level Curve for Exercise 2

(b) Use classic implicit differentiation or some other method to find an equation for the line *L* tangent to *C* at **P**.
Solution: Differentiating $2x^2 + 3y^2$ =98 with respect to *x*, treating *y* as an unspecified function of *x*, we have $4x + 6y\,\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{2x}{3y}$. Then the slope of the tangent line at (5,4) is $-\frac{2}{3} \times \frac{5}{4} = -\frac{5}{6}$. An equation for the tangent line is $y - 4 = -\frac{5}{6}(x - 5)$.
(c) Determine the gradient vector **v** of *f* at **P**.
Solution: $\nabla f(x, y) = \left(f_x(x, y), f_y(x, y)\right) = (4x, 6y)$.Thus **v** = $\nabla f(5,4) = (20,24)$.
(d) Show that this gradient vector is orthogonal to any vector lying along *L*.
Solution: The slope of this gradient vector is $\frac{24}{20} = \frac{6}{5}$ which is negative reciprocal of $-\frac{5}{6}$ so the vector **v** is orthogonal to the tangent line *L*.

2. Let $g(x, y) = 2x^2 - 3y^2$ be a real-valued function defined on the plane.
   (a) Let **P** be the point (5,4) and *C* the level curve of *g* containing P. Identify the nature of *C:* Is it a Circle? Parabola? Pair of Lines? Sketch a picture of *C.*
   Solution: $g(5,4) = 2(5^2) - 3(4^2) = 50 - 48 = 2.$ Now $2x^2 - 3y^2$ =2 is the equation of an **hyperbola.** Above right is a graph of the level curve.
   (b) Use classic implicit differentiation or some other method to find an equation for the line *L* tangent to *C* at **P**.
   Solution: Differentiating $2x^2 - 3y^2$ =2 with respect to *x*, treating *y* as an unspecified function of *x*, we have $4x - 6y\,\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = \frac{2x}{3y}$. Then the slope of the tangent line at (5,4) is $\frac{2}{3} \times \frac{5}{4} = \frac{5}{6}$.
   An equation for the tangent line is $y - 4 = \frac{5}{6}(x - 5)$.
   (c) Determine the gradient vector **v** of *g* at **P**.
   Solution: $\nabla f(x, y) = \left(f_x(x, y), f_y(x, y)\right) = (4x, -6y)$.Thus **v** = $\nabla f(5,4) = (20, -24)$.
   (d) Show that this gradient vector is orthogonal to any vector lying along *L*.

Solution: The slope of this gradient vector is $\frac{-24}{20} = \frac{-6}{5}$ which is negative reciprocal of $\frac{5}{6}$ so the vector **v** is orthogonal to the tangent line *L*.

3. (Williamson and Trotter) A spaceship traveling in the plane along a path such that at time $t \geq 0$, the ship is at position $g(t) = (3t^2, t^3)$. The intensity of gamma radiation at the point (x,y) in the plane is $I(x, y) = x^2 - y^2$,
wherever $I(x, y) \geq 0$. Describe fully, using a labeled graph where appropriate, the following:
(a) The level curve of I the ship is on at $t = 1$.
Solution: $g(1) = (3,1)$ is the position at $t = 1$ where $I(3,1) = 3^2 - 1^2 = 8$.
(The level curve is the hyperbola $x^2 - y^2 = 8$
(b) The path of the ship for $t \geq 0$.
Solution: The rectangular coordinates of the spaceship's position at time t satisfy $x = 3t^2, y = t^3$. For $t \geq 0$, we have $t = \left(\frac{x}{3}\right)^{1/2}$ so $y = \left(\frac{x}{3}\right)^{3/2}$. The graph of this relationship is the path of the spaceship with $t = 0$ corresponding to $(x,y) = (0,0)$; we show it below:



(c) The gradient vector of I at the ship's position when $t = 1$.
Solution: Gradient vector of I is $\nabla f(x, y) = \left(f_x(x, y), f_y(x, y)\right) = (2x, -2y)$ so

$$\nabla f(3,1) = (2 \times 3, -2 \times 1) = (6, -2)$$

(d) The ship's velocity vector at $t = 1$.
Solution: The velocity vector at time t is $g'(t) = (6t, 3t^2)$ and hence the velocity at t=1 is (6, 3).

(e) The time – if there is one – when the ship stops increasing its radiation risk and begins its race to safety. Does its course become more dangerous later on?
Solution: The radiation experienced by the spaceship at time t is represented by the composite function $I(g(t)) = I(3t^2, t^3) = (3t^2)^2 - (t^3)^2 = 9t^4 - t^6 = t^4(9 - t^2)$
which is non-negative for $0 \leq t \leq 3$. The rate at which the radiation is changing is $(I(g(t)))' = 36t^3 - 6t^5 = 6t^3(6 - t^2)$ which is positive for $0 < t < \sqrt{6}$ and negative for $t > \sqrt{6}$. The maximum radiation occurs at $t = \sqrt{6}$ where level is $\left(\sqrt{6}\right)^4 \left(9 - \left(\sqrt{6}\right)^2\right) = (36)(9 - 6) = 108$. The spaceship begins its race to safety at $t = \sqrt{6}$ from which time it experiences less and less radiation until it drops to zero at $t = 3$. The graph of the radiation as function of time is below: