MATH 223

Some Notes on Assignment 30

Exercises 3, 4, 7 11, and 14 of Chapter 8.

3: Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = (1, 2, 3)$. Show that divergence and curl of $\mathbf{F}$ is zero.

Solution: Divergence is $1_x + 2_y + 3_z = 0 + 0 + 0 = 0$.

$$\text{Curl } = \nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & 3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 1 & 3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 1 & 2 \end{pmatrix} \mathbf{k} = (3_y - 2_z)\mathbf{i} - (3_x - 1_z)\mathbf{j} + (2_x - z_y)\mathbf{k} = (0, 0, 0)$$

4: Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = (x^2, xz, yz)$. Find the divergence and curl of $\mathbf{F}$ at (3,2,1) and (1,2,3).

Solution: Divergence is $(x^2)_x + (xz)_y + (yz)_z = 2x + 0 + y = 2x + y$.

$$\text{Curl } = \nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xz & yz \end{pmatrix} = \det \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2 & yz \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 & xz \end{pmatrix} \mathbf{k}$$

$$= ((yz)_y - (xz)_z)\mathbf{i} - ((yz)_x - x_z^2)\mathbf{j} + ((xz)_x - (yz)_y)\mathbf{k} = (z - x)\mathbf{i} - (0 - 0)\mathbf{j} + (z - 0)\mathbf{k} = (z - x, 0, z)$$

Curl at (3,2,1) = (1 -3,0,1) = (-2,0,1); Curl at (1,2,3) = (3 -1,0,3) = (2,0,3).

7: Show that the divergence and curl of any constant vector field are zero.

Solution: The result follows from the fact that every partial derivative of every component of the vector field will be 0.

11: A vector field is **solenoidal** if its divergence is identically 0. Show that the curl $\mathbf{F}$ is always solenoidal.

Solution: This result follows from the theorem that the divergence of the curl is always 0.

14: Let $\mathbf{F} = \nabla \arctan(y/x)$ Show that curl $\mathbf{F}$ is identically zero and div $\mathbf{F}$ is also identically 0.

Solution: Computing the gradient of $\arctan(y/x)$, we have

$$\mathbf{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Thus

$$div\, \mathbf{F} = \left(\frac{-y}{x^2 + y^2} \right)_x + \left(\frac{x}{x^2 + y^2} \right)_y = \frac{2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0.$$

$$\text{The curl is } \left(\frac{x}{x^2 + y^2} \right)_x - \left(\frac{-y}{x^2 + y^2} \right)_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0.$$