MATH 223
Some Notes on Assignment 23
Exercises A, B, C and D

A Give a careful argument that the limit of a function whose values are all non-negative can not be negative.
Solution: Suppose to the contrary that there is a function f and a point $\mathbf{a}$ such that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ but

$$\lim_{\mathbf{x}\to\mathbf{a}} f(x) = B$$

where B is a negative number.
Let $r = |B|/2$ and $\mathcal{N}$ be the r-neighborhood of B. Note that this neighborhood is just the interval from $B - |B|/2$ to $B + |B|/2$. [Example: if B = -4, then $\mathcal{N}$ is open interval $-6 < t < -2$]. Observe that every number in $\mathcal{N}$ is negative. Since the limit exists, there is some neighborhood $\mathcal{M}$ of $\mathbf{a}$ such that if x is in $\mathcal{M}$ (and different from $\mathbf{a}$), then $f(x)$ must lie in $\mathcal{N}$. That would make $f(x)$ negative, contradicting our assumption that all values of the function are non-negative.

B Verify that Leibniz's Rule is correct for each of the following:

1. $\int_3^9 (x^2 + y^2)\, dy$
   Solution: $F(x) = \int_3^9 (x^2+y^2)\, dy = \left[x^2y + \frac{y^3}{3}\right]_{y=3}^{y=9} = \left(9x^2 + \frac{9^3}{3}\right) - \left(3x^2 + \frac{3^3}{3}\right) = 6x^2 + \frac{9^3}{3} - \frac{3^3}{3}$ so $F'(x) = 12x$. Leibniz's Rules gives $F'(x) = \int_3^9 (x^2 + y^2)_x\, dy = \int_3^9 2x\, dy = [2xy]_{y=3}^{y=9} = 18x - 6x = 12x$.

2. $\int_2^5 x + y^{-3}\, dy$
   Solution:. Let $F(x) = \int_2^5 (x + y^{-3})\, dy$.
   Then direct calculation gives $F(x) = \left[xy - \frac{1}{2}y^{-2}\right]_{y=2}^{y=5} = (5x - \frac{1}{50}) - (2x - \frac{1}{8}) = 3x + \frac{21}{200}$
   so $F'x) = 3$.
   Leibniz's Rule gives $F'(x) = \int_2^5 (x + y^{-3})_x\, dy = \int_2^5 1\, dy = 3$

3. $F(x) = \int_1^e xy + \ln y - \arctan y\, dy$
   Solution: A direct calculation involves using integration by parts to show that $\int \ln y\, dy = y \ln y - y$ and integration by parts again to show that $\int \arctan y\, dy = y \arctan y - \frac{\ln(1+y^2)}{2}$
   Thus

$$\int_1^e (xy + \ln y - \arctan y\, dy = \left[\frac{xy^2}{2} + y \ln y - y - y \arctan y + \frac{\ln(1+y^2)}{2}\right]_{y=1}^{y=e}$$

   which equals
   $\frac{e^2x}{2} - e \arctan e + \frac{\ln(1+e^2)}{2} + \frac{x}{2} - \frac{\pi}{4} + \frac{\ln 2}{2} - 1$ whose derivative with respect to x is $\frac{e^2-1}{2}$
   Leibniz's Rule gives $F'(x) = \int_1^e (xy + \ln y - \arctan y)_x\, dy = \int_1^e y\, dy = \left[\frac{y^2}{2}\right]_1^e = \frac{e^2-1}{2}$

C Use Leibniz's Rule to determine $F'(x)$ if $F(x) = \int_0^1 2\frac{\sin xy}{y}\, dy$.
Solution: By Leibniz's Rule

$$F'(x) = \int_0^{12} \left(\frac{\sin xy}{y}\right)_x dy = \int_0^{12} \frac{1}{y} \cos(xy) y\, dy = \int_0^{12} \cos(xy)\, dy = \left[\frac{1}{x}\sin(xy)\right]_{y=0}^{y=12} = \frac{\sin 12x}{x}$$

D Use Leibniz's Rule to determine $G'(y)$ if $G(y) = \int_{-5}^5 \frac{1-e^{-xy}}{x}\, dx$.
Solution: By Leibniz's Rule

$$G'(y) = \int_{-5}^5 \left(\frac{1-e^{-xy}}{x}\right)_y dx = \int_{-5}^5 \frac{1}{x}(-e^{-xy})(-x)\, dx = \int_{-5}^5 e^{-xy}\, dx = \left[\frac{e^{-xy}}{-y}\right]_{x=-5}^{x=5} = \frac{e^{5y} - e^{-5y}}{y}$$