MATH 223

Corrected Notes on Assignment 20

Exercises 52 and 53 pf Chapter 5; Exercises 1ac, 2 ac, 3, and 4d of Chapter 6.

52: Replace x with $r\cos\theta$ and y with $r\sin\theta$, noting that $x^2+y^2=r^2$.

(a) $5x = 2(x^2+y^2)(4+3xy)$ becomes $5r\cos\theta = 2(r^2)(4+3r\cos\theta r\sin\theta)$ which simplifies to $\cos\theta = 2r(4+3r\cos\theta r\sin\theta)$.

(b) $y^2 - 2\frac{y}{x} = 12 - 7x^2$ becomes $r^2\sin^2\theta - 2\frac{r\sin\theta}{r\cos\theta} = 12 - 7(r^2\cos^2\theta)$ which we can simplify to $r^2\sin^2\theta - 2\tan\theta = 12 - 7r^2\cos^2\theta$ or

$$r^2 = \frac{12+2\tan\theta}{\sin^2\theta + 7\cos^2\theta}$$

53: (a) $r\cos\theta = 8$ becomes $x=8$ so graph is vertical line through (8,0).

(b) $r\sin\theta = -4$ becomes $y=-4$ so graph is horizontal line through (0,-4).

(c) Multiply $r = a\cos\theta$ by r to get $r^2 = ar\cos\theta$ which is the same as $x^2+y^2 = ax$ or $(x-a/2)^2+y^2 = (a/2)^2$ (completing the square in x which is the equation of a circle with center $(a/2, 0$ and radius $a/2$.

(d) Multiply $r = b\sin\theta$ by r to obtain $r^2 = br\sin\theta$ or $x^2+y^2 = by$. Hence $x^2+y^2-by=0$. Complete the square in $y: x^2+y^2-by+(b/2)^2 = (b/2)^2$ or $x^2+(y-b/2)^2 = (b/2)^2$ which is an equation for the circle of radius $b/2$, centered at $(0, b/20)$.

Exercises from Chapter 6

1: For each of the described regions R, represent $\iint_R f(x,y)$ as an iterated integral or integrals using vertical strips and then using horizontal strips: (a) $0 \le x \le 3, 2-x \le y \le 2+x$ and (c) $0 \le x \le 1, x^2 \le y \le x$

Solution: 1. a) If x is bounded above and below by 0 and 3 and y by $2+x$ and $2-x$ then the area being integrated over is a triangle with vertices $(0,2)$, $(3,-1)$, and $(3,5)$. If we integrate with respect to y first to create vertical strips, each strip runs from the line $y=2-x$ to $y=2+x$. The corresponding iterated integral is

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=2-x}^{y=2+x} f(x,y)\,dy\,dx.$$

If we want to integrate with respect to x first to create horizontal strips there will be two different types of strips. When y is between -1 and 2, the horizontal strips are bounded by the lines $x=y-2$ and $x=3$. When y is between 2 and 5 the horizontal strips are bounded by $x=y-2$ and $x=3$. To integrate with respect to x and then y we must then use two separate iterated integrals.

$$\iint_R f(x,y)\,dA = \int_{y=-1}^{y=2}\int_{x=y-2}^{x=3} f(x,y)\,dx\,dy + \int_{y=2}^{y=5}\int_{x=y-2}^{x=3} f(x,y)\,dx\,dy.$$

c) The region described by the inequalities is the area between the lines $y=x$ and $y=x^2$ where x is in the interval $[0,1]$. Notice that this set is both x-simple and y-simple, so integrating in either order will require only one iterated integral. If we wish to integrate with respect to y first, the vertical strips will be bounded below by $y=x^2$ and above by $y=x$. The resulting iterated integral is

$$\iint_R f(x,y) = \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} f(x,y)\,dy\,dx.$$

If we wish to integrate with respect to x first we must rewrite the bounds for y so that they are bounds for x in terms of y. If $y \le x$ then $x \ge y$. If $y \ge x^2$ then $x \le \sqrt{y}$. The bounds for y will now be the same as the bounds for x in the first integral: $0 \le y \le 1$.

$$\iint_R f(x,y) = \int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} f(x,y)\,dx\,dy.$$

2: Sketch the region of integration described by each of the iterated integrals below and write the appropriate iterated integral with the order of integration reversed: (a) $\int_1^3 \int_0^{6-2x} f(x,y)\,dy\,dx$ and (c) $\int_0^2 \int_{1+y/2}^2 f(x,y)\,dx\,dy$

Solution: 2. a) The first integral here, the inner integral, integrates $f(x,y)$ as y runs from 0 to $6-2x$. The area of integration then has one set of boundaries given by $0 \leq y \leq 6-2x$. The outer integral integrates $\int f(x,y)\,dy$ as x runs from 1 to 3. The second set of constraints of our area of integration is then $1 \leq x \leq 3$.

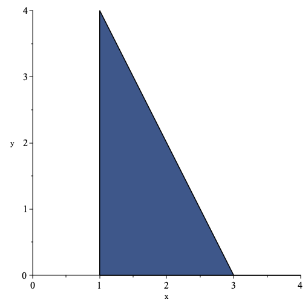

Figure 1: The area of integration for $f(x,y)$.

If we would like to exchange the order of integration, we must find boundaries for x in terms of y and then constant boundaries for y. Recall from the chapter that integrating with respect to x first will leave a function g of y which has $g(y)$ equal $f(x,y)$ integrated over a horizontal line running across the area of integration at a particular value of y. If we draw horizontal lines across our area we see that, for all values of y, the lower bound is 1 and the upper bound is $3-\frac{y}{2}$. The inner integral we are looking for is then

$$\int_{x=1}^{x=3-\frac{y}{2}} f(x,y)\,dx.$$

Of all points (x,y) in the set $1 \leq x \leq 3-\frac{y}{2}$, the only points we would like to integrate over are those for which $0 \leq y \leq 4$. The entire iterated integral is then

$$\int_{y=0}^{y=4} \int_{x=1}^{x=3-\frac{y}{2}} f(x,y)\,dx\,dy.$$

c) The inner integral here has bounds $1+\frac{y}{2} \leq x \leq 2$. Rewriting this first inequality to find y in terms of x we have $y \leq 2x-2$. If y is bound by 0 and 2 then the entire area of integration is the triangle bound by the lines $y = 2x-2$, $y = 0$, and $x = 2$.

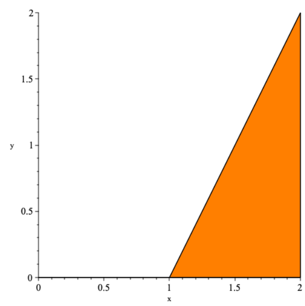

Figure 2: The area of integration for $f(x,y)$.

The given iterated integral integrates first with respect to x and then with respect to y. That is, the given integral uses the horizontal line method. If we use the vertical line method we can see that each vertical line for a given value of x has a lower bound of 0 and an upper bound of $2x-2$. Furthermore, the vertical lines run from $x = 1$ to $x = 2$. The iterated integral in reverse order is then

$$\int_{x=1}^{x=2} \int_{y=0}^{y=2x-2} f(x,y)\,dy\,dx$$

3: Evaluate each of the iterated integrals

1. $\int_0^4 \int_0^y y^2\, dx\, dy$
2. $\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 5\, dx\, dy$

Solution: a) i) If we wish to evaluate the given integral, we must first integrate y^2 with respect to x. Treating y as a constant and using regular differentiation rules we have

$$\int_{y=0}^{y=4} xy^2\, dy = \int_{y=0}^{y=4} y^3\, dy.$$

Integrating again, this time with respect to y we have

$$\int_{y=0}^{y=4} y^3\, dy = \frac{4^4}{4} - 0 = 4^3 = 64.$$

b) Before beginning to evaluate this integral, it will be helpful to sketch the area of integration. Notice that we are integrating over a circle of radius 4 centered at the origin. From the chapter we know that we can factor the constant 5 out of the integral to get

$$\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 5\, dx\, dy = 5 \iint_R 1\, dx\, dy$$

Where R is just the circle of radius 4 centered at the origin. The right hand side of this equation is just five times the area of integration so,

$$\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 5\, dx\, dy = 5\pi 4^2 = 80\pi.$$

ii) Because the area of integration is a circle, it may be easier to evaluate the iterated integral by applying switching to polar coordinates using Jacobi's Theorem. From the text we know that the absolute value of the determinant of the planar transformation to polar coordinates is r. The area of integration described in polar coordinates is $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$. We have

$$\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 5\, dx\, dy = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} 5r\, dr\, d\theta$$

If we factor the 5 out of the right hand side and integrate with respect to r we have

$$5 \int_{\theta=0}^{\theta=2\pi} \frac{4^2}{2} - \frac{0^2}{2}\, d\theta = 5 \int_{\theta=0}^{\theta=2\pi} 8\, d\theta.$$

Integrating with respect to θ and evaluating over the interval $[0, 2\pi]$ we find

$$\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 5\, dx\, dy = 80\pi.$$

4ad:

Solution: Recall from the chapter that the area bounded by two curves is equal to the iterated integral of the function $f(x, y) = 1$ over the same area provided that the iterated integral exists in some order. To find the area of the region we then need to find an valid iterated integral.

a) Begin by sketching the region bounded by the two curves.

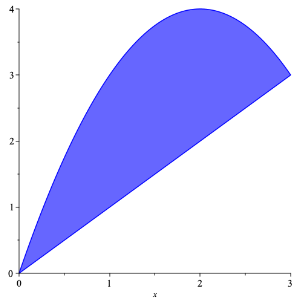

Figure 3: The area bounded by $y = x$ and $y = 4x - x^2$.

This region is both x-simple and y-simple, so either order of integration should be straightforward. If we draw vertical lines covering the area, a line for a particular value of x will have a lower bound of $y = x$ and an upper bound of $y = 4x - x^2$. The values of x range from 0 to 3. The iterated integral of interest is then

$$\int_{x=0}^{x=3} \int_{y=x}^{y=4x-x^2} 1\, dy\, dx.$$

Integrating with respect to y and evaluating over at the upper and lower bounds we get

$$\int_{x=0}^{x=3} \int_{y=x}^{y=4x-x^2} 1\, dy\, dx = \int_{x=0}^{x=3} x^2 - x\, dx.$$

Integrating the right hand side of this last equation with respect to x and evaluating at 0 and 3 leaves

$$\int_{x=0}^{x=3} \int_{y=x}^{y=4x-x^2} 1\, dy\, dx = \frac{3^3}{3} - \frac{3^2}{2} = \frac{9}{2}.$$

d)

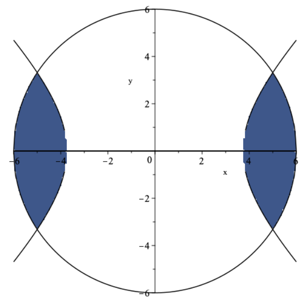

Figure 4: The area bounded by $x^2 - y^2 = 14$ and $x^2 + y^2 = 36$.

Notice here that the area we would like to find is actually two separate areas of equivalent size. Furthermore, both of these smaller regions are split in half by the x axis. This allows us to solve for the total area by solving for one quarter and multiplying by 4. For the sake of simplicity, we will solve for the area of the quarter which lies completely in the first quadrant. Because we are dealing with an area described by a circular boundary, converting to polar coordinates and applying Jacobi's Theorem will give us simpler expressions to integrate. In polar coordinates, the circular curve is described by $r = 6$ and the hyperbolic curve is

$$r^2 \cos^2\theta - r^2 \sin^2\theta = 14 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 14$$

Applying the double angle formula to the right hand side we have

$$r^2 \cos^2\theta - r^2 \sin^2\theta = 14 \Rightarrow r = \sqrt{\frac{14}{\cos 2\theta}}.$$

In polar coordinates, r has minimum of $\sqrt{\frac{14}{\cos 2\theta}}$ and a maximum value of 6 while θ ranges from 0 to some constant. Note that θ achieves its maximum when the boundary curves for r are equivalent. That is, when $6 = \sqrt{\frac{14}{\cos 2\theta}}$, theta achieves its maximum value for the region. Solving this equation for θ reveals the upper bound to be $\theta = \frac{\arccos \frac{7}{18}}{2}$. Now that we have bounds for the region in polar coordinates, we can set up equivalent iterated integrals using Jacobi's theorem.

$$\iint_R 1dA = \int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \int_{r=\sqrt{\frac{14}{\cos 2\theta}}}^{r=6} 1 \cdot r\,dr\,d\theta$$

Integrating the right hand side with respect to r gives

$$\iint_R 1dA = \int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} |_{r=\sqrt{\frac{14}{\cos 2\theta}}}^{r=6} \frac{r^2}{2}\,d\theta$$

$$\iint_R 1dA = \int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} 18\,d\theta \quad - \quad \int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \frac{7}{\cos 2\theta}\,d\theta.$$

Integrating 18 with respect to θ is no problem; however, the second integral is more tricky. To make it slightly more simple, we may right it as

$$\int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \frac{7}{\cos 2\theta}\,d\theta = 7\int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \sec(2\theta)\,d\theta.$$

Now let $u = 2\theta$ and integrate using u substitution. (The general integral of $\sec x$ is $\ln|\sec x + tanx|$.

$$7\int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \sec(2\theta)\,d\theta = \frac{7}{2} \quad |_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \ln|\sec 2\theta + \tan 2\theta|.$$

Substituting the right hand side of this last equation into our original integral we have

$$\iint_R 1dA = \int_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} 18\,d\theta \quad - \frac{7}{2} \quad |_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \ln|\sec 2\theta + \tan 2\theta|.$$

$$\iint_R 1dA = |_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} 18\theta - \frac{7}{2} \quad |_{\theta=0}^{\theta=\frac{\arccos \frac{7}{18}}{2}} \ln|\sec 2\theta + \tan 2\theta|$$

Now all we have to do is evaluate each term at the boundaries to find

$$\iint_R 1dA \approx 4.95$$

Multiplying this value by 4 to account for the area in each quadrant reveals that the total area is 19.8.