MATH 223

*Some Notes on Assignment 17*

Exercises 13, 15, 16, and 17 in Chapter 5; Problem A

**13:** For each of the following implicitly defined functions, determine $\frac{dy}{dx}$ at the given points:
(a) $xy + 2 = 0$ at $(x, y) = (2, -1)$
(b) $xe^y + ye^x = 0$ at the origin (0,0).
(c) $x^2 + y^2 = 25$ at $(x, y) = (3, -4)$.

*Solution:* (a) If we find $y$ in terms of $x$ we can apply the normal rules of differentiation of a single variable to solve for $\frac{dy}{dx}$ at $(2, -1)$

$$xy + 2 = 0 \Rightarrow y = \frac{-2}{x} \Rightarrow \frac{dy}{dx} = \frac{2}{x^2}$$

$$\frac{dy}{dx}(2) = \frac{1}{2}$$

Instead of rewriting the initial equation, we may also use Implicit differentiation and the Chain Rule to solve for $\frac{dy}{dx}$.

$$xy + 2 = 0 \Rightarrow 0 = y + (x)\frac{dy}{dx}$$

Examining this second equation at the point $(2, -1)$ we have

$$\frac{dy}{dx} = \frac{-y}{x} = \frac{1}{2}$$

Finally, we can also solve for $\frac{dy}{dx}$ by finding a normal vector to the level curve $f(x, y) = xy + 2 = k$ with $k = 0$. The gradient of $f$ is $\nabla f = (y, x)$. At $(2, -1)$ the gradient is then $\nabla f = (-1, 2)$. From the Normal Vectors section of the text, we know that this gradient vector is perpendicular to the tangent vector of the level set at $(2, -1)$. Thus, the tangent vector must have slope $\frac{1}{2}$.

(b) If we are interested in finding $\frac{dy}{dx}$ for the equation $xe^y + ye^x = 0$, then we have two options. Notice that there are two $y$ terms, one of which is in an exponent, so solving for $y$ explicitly in terms of $x$ is difficult. Instead, we can either implicitly differentiate or use the gradient method.
Implicitly differentiating the equation using the Chain Rule and the Product Rule we get

$$e^y + xe^y\frac{dy}{dx} + \frac{dy}{dx}e^x + ye^x = 0$$

$$\frac{dy}{dx} = \frac{-(ye^x + e^y)}{xe^y + e^x}$$

Examining this last equation at the origin we find $\frac{dy}{dx} = -1$.
Recall that the gradient vector of the function $f(x, y) = xe^y + ye^x$ will be perpendicular to the line tangent to the level curve $xe^y + ye^x = 0$ at the origin. Thus, we can find the slope of this level curve by finding the slope of a vector that is perpendicular to it.

$$\nabla f = (e^y + ye^x, xe^y + e^x)$$

$$\frac{dy}{dx} = \frac{-(e^y + ye^x)}{xe^y + e^x}$$

Evaluating at the origin we have $\frac{dy}{dx} = -1$.

(c) Solving the equation $x^2 + y^2 = 25$ for $y$ results in two possible answers for $y$: $y = \pm\sqrt{25 - x^2}$. We are interested in the point $(3, -4)$, so the form of this equation we should differentiate is $y = -\sqrt{25 - x^2}$. Differentiating with respect to $x$ we get

$$\frac{dy}{dx} = \frac{x}{\sqrt{25 - x^2}}.$$

At the point $(3, -4)$ we have $\frac{dy}{dx} = \frac{3}{4}$.
If we implicitly differentiate $x^2 + y^2 = 25$ we find

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

At the origin, $\frac{dy}{dx} = \frac{3}{4}$.
Finally, the gradient of $f(x, y) = x^2 + y^2$ will be perpendicular to any line tangent to the level set $x^2 + y^2 = 25$. Thus, $\frac{dy}{dx}$ is equal to $-1$ times the $x$ component of the gradient divided by the $y$ component of te gradient.

$$\nabla f = (2x, 2y) \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Evaluated at the point $(3, -4)$ we we have $\frac{dy}{dx} = \frac{3}{4}$.

**15.** Use Solution Method B to find the slope of the tangent line to the curve in Example 2 at the point (2,3). Note that $\frac{dxy}{dx} = y + x\frac{dy}{dx}$.
*Solution:* 15. If we implicitly differentiate with respect to $x$ using the Chain and Product Rules we have

$$\frac{d}{dx}(2x^3 + 2y^3 - 9xy) = \frac{d}{dx}(16)$$

$$6x^2 + 6y^2\frac{dy}{dx} - 9(y + x\frac{dy}{dx}) = 0$$

$$\frac{6x^2 - 9y}{9x - 6y^2} = \frac{dy}{dx}.$$

Evaluating the last expression at the point $(2, 3)$ gives $\frac{dy}{dx} = \frac{1}{12}$.

**16**. Find the slope of the tangent line to the curve in Example 2 at the point (2,-3)
*Solution:* Applying the same expression for $\frac{dy}{dx}$ as was found in the previous section we substitute $(2, -3)$ for $(x, y)$ to find $\frac{dy}{dx}f(2, -3) = -\frac{17}{12}$.

**17.** Show that the minimum value of a real-valued function $f$ is the maximum value of $-f$ so that you can solve any minimization problem by solving a simply related maximization problem.

*Solution:* Suppose that a function $f$ has an absolute minimum at some point $x_0$. This implies that for all $x$ for which $f$ is defined, $f(x_0) \leq f(x)$. The function $-f$ is defined on the same set as $f$, so if we multiply the above inequality by $-1$ we get $-f(x_0) \geq -f(x)$

for all $x$ on which $-f$ is defined. The point $x_0$ is then an absolute maximum of the function $-f$.

**Problem A** The equation $x^2/4 + y^2/9 + z^2 - 6 = 0$ defines $z$ implicitly as a function $z = f(x, y)$ near the point $\mathbf{P}$ where $(x = 2, y = 3, z = -2)$. The graph of the function $f$ is a surface. Find its tangent plane at $\mathbf{P}$.
*Solution:* Taking the partials of $x^2/4 + y^2/9 + z^2 - 6 = 0$ first with respect to $x$ and then with respect to $y$ yields

$$\begin{array}{c|c}
\frac{2x}{4} + 0 + 2zf_x(x, y) = 0 & 0 + \frac{2y}{9} = 2zf_y(x, y) = 0 \\
\text{Substitute } x = 2, y = 3, z = -2 & \text{Substitute } x = 2, y = 3, z = -2 \\
1 - 4f_x(2, 3) = 0 & \frac{2}{3} - 4f_y(2, 3) = 0 \\
f_x(2, 3) = \frac{1}{4} & f_y(2, 3) = \frac{1}{6}
\end{array}$$

Thus an equation for the tangent plane is $z = -2 + \frac{1}{4}(x - 2) + \frac{1}{6}(y - 3)$.