MATH 223

Some Notes on Assignment 13

Exercises 30, 31, and 32 in Chapter 4. (Old 23, 24, 25)

For each of the following vector-valued functions, Find the Jacobian matrix and the determinant of the Jacobian matrix and describe the set of points at which the Jacobian matrix is invertible

23. $f(x,y) = \left(\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

Solution: If $f(x,y) = (f_1(x,y), f_2(x,y))$ and f_{1x}, f_{2x}, f_{1y}, and f_{2y} are the partial derivatives of each component function, then the Jacobian matrix of f is

$$\mathbf{M} = \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = \begin{pmatrix} \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \\ \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \end{pmatrix}.$$

Using the formula for a 2 by 2 matrix we find that the determinant of $\mathbf{M}$ is

$$\det \mathbf{M} = \frac{1}{(x^2+y^2)^4}(4x^2y^2 - (x^2-y^2)(y^2-x^2)),$$

$$\det \mathbf{M} = \frac{1}{(x^2+y^2)^4}(x^2+y^2)^2 = \frac{1}{(x^2+y^2)^2}.$$

The determinant of the Jacobian matrix is then non-zero and defined everywhere except for the origin of the xy plane; thus, it is invertible everywhere except the origin.

24: $g(x,y) = (y\cos x, y\sin x)$

Solution: If $g(x,y) = (g_1(x,y), g_2(x,y))$ and g_{1x}, g_{2x}, g_{1y}, and g_{2y} are the partial derivatives of each component function, then the Jacobian matrix of g is

$$\mathbf{M} = \begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix} = \begin{pmatrix} -y\sin x & \cos x \\ y\cos x & \sin x \end{pmatrix}.$$

Using the formula for a 2 by 2 matrix we find that the determinant of $\mathbf{M}$ is

$$\det \mathbf{M} = -y\sin^2 x - y\cos^2 x = -y(\sin^2 x + \cos^2 x) = -y.$$

The determinant of the Jacobian matrix $\mathbf{M}$ is non zero if and only if y is nonzero. The matrix is then invertible everywhere except the line $y = 0$.

25: $h(x,y,z) = (yz, xz, xy)$

Solution: If $h(x,y,z) = (yz, xz, xy)$ then the Jacobian matrix of h is

$$\mathbf{M} = \begin{pmatrix} h_{1x} & h_{1y} & h_{1z} \\ h_{2x} & h_{2y} & h_{2z} \\ h_{3x} & h_{3y} & h_{3z} \end{pmatrix} = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}.$$

Evaluating the determinant of $\mathbf{M}$ by expanding along the first row we find

$$\det \mathbf{M} = zxy + yzx = 2xyz.$$

The determinant is non zero everywhere that x, y, and z are nonzero. The Jacobian matrix is invertible everywhere except for the coordinate planes.

Problem A: Show that the Mean Value formula of Theorem 4.4.1 also takes the form $\frac{f(\mathbf{y})-f(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} = f_{\mathbf{u}}(\mathbf{x_0})$ where the unit vector is $\mathbf{u} = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$

Solution: Assuming $\mathbf{x} \neq \mathbf{y}$, we can divide both sides of the conclusion of the theorem by the nonzero scalar $|\mathbf{y}-\mathbf{x}|$ to write

$$\frac{f(\mathbf{y})-f(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} = \frac{\nabla f(x_0)\cdot(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} = \nabla f(x_0)\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$$

since $\frac{\mathbf{v}\cdot\mathbf{w}}{\alpha} = \mathbf{v}\cdot\left(\frac{\mathbf{w}}{\alpha}\right)$ for vectors $\mathbf{v}$ and $\mathbf{w}$ and scalar α. Lastly, note that if $\mathbf{u} = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$, then the right side of the last equation is the definition of $f_{\mathbf{u}}(\mathbf{x_0})$.

Problem B: The Mean Value Theorem may fail for vector-valued functions. Consider the function $f(t) = (\sin t, \sin 2t), 0 \leq t \leq \pi$. Show that there is no point x_0 between x and y at which $\frac{f(y)-f(x)}{|y-x|} = f'(x_0)$ where $x = 0$ and $y = \pi$

Solution: $f(y) = f(\pi) = (0,0), f(x) = f(0) = (0,0)$ so $\frac{f(y)-f(x)}{|y-x|} = \frac{1}{\pi}(0,0) = (0,0)$ but $f'(x_0) = (\cos x_0, 2\cos 2x_0)$. Note that $\cos x_0 = 0$ when x_0 is an odd multiple of $\pi/2$ in which case $2x_o$ is an odd multiple of π which has $\cos 2x_0 = -1$.