MATH 223

Some Notes on Assignment 5

Chapter 2: Exercises 45, 51, Chapter 3: 4, 6 and Problem A

45: For Example 1 in our discussion of projectile motion, if the bat hits the ball at a height of 4 feet and at an angle of $\pi/6$, what is the minimum value of the initial speed that will guarantee that the ball will pass over the Green Monster?
Solution: Following the lead of the first example in the Projectile Motion section, we can write the position of the ball as $\mathbf{p}(t) = (x(t), y(t)) = (v \cos \frac{\pi}{6} t, v \sin \frac{\pi}{6} t - 16t^2 + 4) = (v\frac{\sqrt{3}}{2}t, \frac{v}{2}t - 16t^2 + 4)$. Now if we solve $x(t) = 310$, we can find the time at which the ball will reach the Green Monster.

$$x(t_0) = 310 = v(\frac{\sqrt{3}}{2})t_0 \Rightarrow \frac{620}{v\sqrt{3}} = t_0$$

If we substitute t_0 into the coordinate function $y(t)$, we can solve for v_0 such that $y(t_0) = 37.166$. At any initial speed greater than v_0 the ball will pass over the wall for a home run.

$$y(t_0) = 37.166 = (\frac{v}{2})(\frac{620}{v\sqrt{3}}) - (16)(\frac{620}{v\sqrt{3}})^2 + 4$$

$$v^{-1} = \sqrt{(\frac{310}{\sqrt{3}} - 33.166)(\frac{1}{16})} * \frac{\sqrt{3}}{620}$$

$$v = 118.6$$

If the initial speed of the ball is greater that 118.6 feet per second, the batter will hit the ball over the Green Monster for a home run.

51: How deep is a well if you drop a stone into it and hear the splash of water 4 seconds later? Assume the speed of sound is 1125 feet/second.
Solution: We can simplify this problem by breaking it into two events, each of which can be described by projectile motion. First, the stones is dropped some unknown depth into the well. Then distance of the stone after t seconds is given by $x(t) = 16t^2$. If d is the depth of the well then, then $t_1 = \frac{\sqrt{d}}{4}$ is the time it takes for the stone to reach the water. Once the stone hits the water, the noise from the splash rushes up the well with speed 1125 feet per second. If t_2 is the time between the splash occurring and the noise being heard then $t_2 = \frac{d}{1125}$. The sum $t_1 + t_2$ is 4 seconds.

$$4 = t_1 + t_2 = \frac{(1125)\sqrt{d} + 4d}{4(1125)}$$

$$0 = 4d + 1125\sqrt{d} - (4)(4)(1125)$$

If we let x = $\sqrt{d}$ we get

$$0 = 4x^2 + 1125x - 1800$$

We can now apply the quadratic formula to find $x = 15.18 \Rightarrow d = 230.4$.

Chapter 3: 4. For the 2 by 2 matrix $M = \begin{pmatrix} x & z \\ 3 & y \end{pmatrix}$, describe the set of points in $\mathbb{R}^3$ where M is not invertible.
Solution The matrix M fails to be invertible precisely when its determinant is 0. Here det $M = xy - 3z$ which is 0 along the surface $z = xy/3$, a two-dimensional set in three-dimensional space (Figure 1). Thus M is invertible on all but a relatively small subset of $\mathbb{R}^3$.

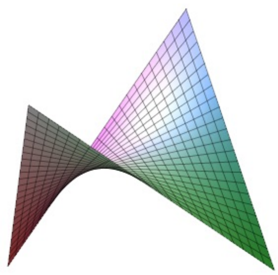


Figure 1

Chapter 3: 6. The function $f(x) = \sqrt{9 - x^2 - y^2}$ is defined where ever $9 - x^2 - y^2 \geq 0$. The domain of f is then a disk of radius 3 centered at the origin of the xy plane (Figure 2). The graph of the function is a surface in $\mathbb{R}^3$ opening downwards over the domain (Figure 3). Cross sections of the graph are semicircles for fixed x and fixed y, circles for fixed z.. Because f is real-valued, the image of f is an interval of $\mathbb{R}$: $[0, 3]$.

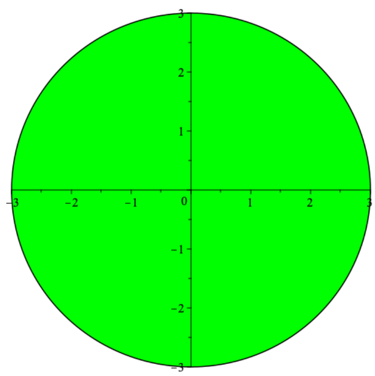
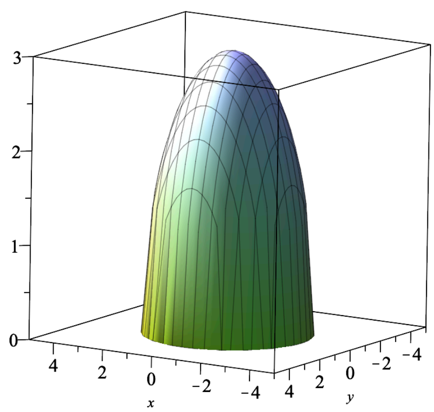


Figure 2



. Figure 3

Problem A: In World War I, Germany bombarded Paris by guns from the unprecedented distance of 75 miles away, shells taking 186 seconds to complete their trajectories. Estimate the angle of elevation at which the gun (called *Big Bertha*) was fired and the maximum height of the trajectory, assuming negligible air resistance. (During a substantial part of the trajectory the altitude was high enough that air resistance was negligible there).

Solution: We'll use feet to measure distance and seconds to measure time.
Let θ be the angle at which the gun is fired.
Let v_0 be the initial speed of the shell.
Let h_{max} be the maximum height of the trajectory.
Let $\mathbf{x}(t)$ be the position vector of the shell.

Then our initial conditions are $\mathbf{x}(0) = (0, 0)$ and $\mathbf{x}'(t) = (v_0 \cos\theta, v_0 \sin\theta)$.
The acceleration vector is $\mathbf{x}''(t) = (0, -32)$.
Integrating with respect to t twice, we have

$$\mathbf{x}(t) = \Big((v_0 \cos\theta)t, -16t^2 + (v_0 \sin\theta)t\Big)$$

The shell's flight time is 186 and its horizontal travel distance is 75 miles = 396,000 feet,
Thus

$$\mathbf{x}(186) = (186v_0 \cos\theta, -16(186)^2 + 186v_0 \sin\theta) = (396000, 0)$$

Equating components, we have

$$v_0 \cos\theta = \frac{396000}{186}, 186v_0 \sin\theta = 16(186)^2$$

which simplifies to

$$v_0 \cos\theta = 2129, v_0 \sin\theta = 2976$$

so that $\tan\theta = \frac{2976}{2129} = 1.3978$ and hence $\theta \approx 54.4^o$

With no air resistance, the shell reaches maximum height haflway through the flight ; that is, $186/2 = 93$ seconds where

$$h_{max} - 16(93)^2 + 2976(93) = 138385 \text{ feet } \approx 26.2 \text{ miles .}$$