CLASS 16

HANDOUTS

NOTES ON Assignment 15
Assignment 16
Sample Exam 2

Announcements

EXAM 2 April Eleven
NO office Hours Today

Topics THIS week

MORE ON Implicit Differentiation
EXTREME VALUES
GRADIENT Fields

GRADIENT FIELDS

If $f: \mathbb{R}^n \to \mathbb{R}^1$, then $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$ is a special type vector field, called a GRADIENT FIELD and f is a POTENTIAL for ∇f.

Example The Vector Field $F: \mathbb{R}^2 \to \mathbb{R}^2$ which is

$$\vec{F}(x,y) = (2x \sin y, \; x^2 \cos y)$$

is a potential for

$$f(x,y) = x^2 \sin y$$

Note: $f_x = 2x \sin y$ $\qquad f_y = x^2 \cos y$

and $f_{xy} = 2x \cos y$ $\qquad f_{yx} = 2x \cos y$

Note: equality of mixed partials

NOT ALL VECTOR FIELDS ARE GRADIENT FIELDS

Example $F(x,y) = (y \cos x, \; x^2 + \sin y)$

If $F = \nabla f$, then

$$f_x = y \cos x \qquad f_y = x^2 + \sin y$$

$$\Rightarrow \quad f_{xy} = \cos x \quad \text{but} \quad f_{yx} = 2x$$

MIXED PARTIALS UNEQUAL.

TRY TO WORK BACKWARDS

Begin with $f_x(x,y) = y \cos x$

Then $\int_x f_x = y \sin x + g(y)$

BUT PARTIAL with respect to y is $x^2 + \sin y$

CAN we pick $g(y)$ so that

$$\sin x + g'(y) = x^2 + \sin y \; ?$$

NO.

EXAMPLE: FIND POTENTIAL f where

$$\nabla f = \left(2x \ln(xy) + x - y^3,\ \frac{x^2}{y} - 3y^2 x\right)$$

① Check equality of mixed partials:

$$f_{xy} = 2x \cdot \frac{1}{xy} \cdot x - 3y^2 = 2x + \frac{1}{y} - 3y^2$$

$$f_{yx} = \frac{2x}{y} - 3y^2$$

② INTEGRATE with respect to y: $\int \frac{x^2}{y} - 3y^2 x \, dy$

$$f \overset{?}{=} x^2 \ln y - y^3 x + H(x)$$

③ Check f_x: $2x \ln y \cancel{+} y^3 + H'(x)$

Need $2x \ln y \cancel{+} y^3 + H'(x) = 2x \ln(xy) + x - y^3$

$$2x \ln y - y^3 + H'(x) = 2x \ln x + 2x \ln y + x - y^3$$

$$H'(x) = 2x \ln x + x$$

so can take $H(x) = x^2 \ln x + C$

④ THUS

$$f(x,y) = x^2 \ln y - y^3 x + x^2 \ln x + C$$

Implicit Differentiation

Example The equations $2x^3y + yx^2 + t^2 = 0$
$x + y + t - 1 = 0$

together define x and y implicitly as functions of t

EACH EQUATION DEFINES A SURFACE IN $\mathbb{R}^3$

Intersection of 2 surfaces is a curve in $\mathbb{R}^3$

which has some parametrization

$$G(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}, \quad \mathbb{R}^1 \longrightarrow \mathbb{R}^3$$

CONSIDER $\mathbb{R}^1 \xrightarrow{G} \mathbb{R}^3 \xrightarrow{F} \mathbb{R}^2$

where $\vec{F}(x,y,t) = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix} = \begin{pmatrix} 2x^3y + yx^2 + t^2 = 0 \\ x + y + t - 1 = 0 \end{pmatrix}$

THEN $F(G(t)) = 0$ for all t.

Differentiate using Chain Rule

$$[F(G(t))]' = \underset{2\times 3}{F'(G(t))} \cdot \underset{3\times 1}{G'(t)}$$

$$\begin{pmatrix} F_{1x} & F_{1y} & F_{1t} \\ F_{2x} & F_{2y} & F_{2t} \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6x^2y + 2yx & 2x^3 + x^2 & 2t \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can write as

$$\begin{pmatrix} 6x^2y + 2xy \\ 1 \end{pmatrix} + \begin{bmatrix} 2x^3 + x^2 & 2t \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we have a particular point (x_0, y_0, t_0) we can solve for x' and y' at this step.

$$\begin{bmatrix} 2x^3+x^2 & 2t \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = -\begin{bmatrix} 6x^2y+2xy \\ 1 \end{bmatrix}$$

MULTIPLY EACH SIDE by INVERSE

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = -\begin{bmatrix} 2x^3+x^2 & 2t \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 6x^2y+2xy \\ 1 \end{bmatrix}$$

Point $t = 1$

$x = -1$

$y = 1$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = -\begin{bmatrix} -2+1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 6-2 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= -\left(-\tfrac{1}{3}\right)\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

MORE GENERALLY

$$\begin{cases} F_1(x, y, t) = 0 \\ F_2(x, y, t) = 0 \end{cases}$$

define x, y implicitly as functions of t

<u>Problem</u>: FIND $x'(t)$ and $y'(t)$.

Set Up: $\mathbb{R}^1 \xrightarrow{\vec{G}} \mathbb{R}^3 \xrightarrow{\vec{F}} \mathbb{R}^2$

Where

$$G(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix} \qquad F(x, y, t) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

THEN $F(G(t)) \equiv 0$

so

$$F'(G(t)) \cdot G'(t) = 0$$

$$(F_x, F_y, F_t) \begin{bmatrix} 1 \\ x' \\ y' \end{bmatrix} = \vec{0}$$

$$F_x + [F_y, F_t]\,[f'(t)] = 0$$

$$f'(t) = -[F_y, F_t]^{-1} F_x$$

$$F' = \begin{pmatrix} F_{1x} & F_{1y} & F_{1t} \\ F_{2x} & F_{2y} & F_{2t} \end{pmatrix}$$

$$F_x \qquad F_y \qquad F_t$$

$(1 \times 3) \cdot 3 \times 1$

$$G(t) = \begin{pmatrix} t \\ \vec{f}(t) \end{pmatrix}$$