

# MATH 226 Differential Equations



Class 35: May 13, 2022



Power Series

# Announcements

**Project 3 Due Today**

**Course Response Forms  
In Class Next Monday  
Bring Laptop/SmartPhone**

**Final Exam  
Friday, May 20: 9 - Noon**



Example

$$y' = y \text{ with } y(0) = 1$$

$$y'(t) = y(t)$$

$$\frac{y'(t)}{y(t)} = 1$$

Integrate Both Sides With Respect to  $t$

$$\ln y = t + C$$

Exponentiate

$$y = Ce^t$$

Use Initial Condition:  $y = 1$  when  $t = 0$

$$1 = Ce^0 = C \times 1 = C$$

$y = e^t$  is the unique solution

*e t*

How Do We Actually Calculate  $e^t$  for a specific  $t$ ?  
In particular, what is the value of  $e^1 = e$ ?



What Is Going On Inside The Calculator?

## Power Series Approach

Assume solution can be represented by a power series

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

Then we have an expression for the solution if we can determine what the constant coefficients  $a_0, a_1, a_2, \dots$  are.

Example: Solve  $y' = y$  with initial value  $y(0) = 1$

With

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \dots$$

we have

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$



Since  $y' = y$  and

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \dots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

we can equate coefficients

$$\begin{aligned} a_1 &= a_0 & 2a_2 &= a_1 & 3a_3 &= a_2 & \dots & na_n &= a_{n-1} \\ a_1 &= a_0 & a_2 &= \frac{a_1}{2} & a_3 &= \frac{a_2}{3} & \dots & a_n &= \frac{a_{n-1}}{n} \end{aligned}$$

$$a_1 = a_0$$

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{3 \times 2} = \frac{a_0}{3!}$$

$$a_4 = \frac{a_3}{4} = \frac{a_0}{4 \times 3!} = \frac{a_0}{4!}$$

...

$$a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n \times (n-1)!} = \frac{a_0}{n!}$$

So the solution of  $y' = y$  is

$$y = a_0 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots \right)$$

Solution to  $y' = y$  with  $y(0) = 1$  has the form

$$y = a_0 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots \right)$$

Evaluating at  $x = 0$  gives  $1 = y(0) = a_0(1 + 0 + 0 + 0 + \dots) = a_0$   
so  $a_0 = 1$  and

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

But we also know the solution is  $y = e^x$ . Thus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

$$P_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$n$	$P_n(1)$
0	1.0
1	2.0
2	2.50000
3	2.666666667
4	2.708333333
5	2.716666667
6	2.718055556
7	2.718253968
8	2.718278770
9	2.718281526
10	2.718281801
11	2.718281826
12	2.718281828

## Example: A Second Order Differential Equation

$$y'' + xy' + y = 0$$

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \dots$$

$$y' = a_1 + 2a_2x + \dots + na_nx^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1}$$

$$xy' = a_1x + 2a_2x^2 + \dots + na_nx^n + (n+1)a_{n+1}x^{n+1} + (n+2)a_{n+2}x^{n+2} + \dots$$

$$y'' = 2a_2 + \dots + n(n-1)a_nx^{n-2} + n(n+1)a_{n+1}x^{n-1} + (n+1)(n+2)a_{n+2}x^n + \dots$$

Coefficient of  $x^n$  in  $y + xy' + y''$  is

$$\begin{aligned} & a_n + na_n + (n+1)(n+2)a_{n+2} \\ &= a_n(1+n) + (n+1)(n+2)a_{n+2} = (n+1)[a_n + (n+2)a_{n+2}] \end{aligned}$$

But all coefficients must be 0. Thus

Our Equation:  $y'' + xy' + y = 0$  has solution

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \dots$$

$$\text{where } a_{n+2} = -\frac{a_n}{n+2}$$

Thus

$$a_2 = -\frac{a_0}{2}, a_4 = -\frac{a_2}{4} = +\frac{a_0}{2 \cdot 4}, a_6 = -\frac{a_4}{2 \cdot 4 \cdot 6}, \dots$$

$$a_3 = \frac{a_1}{3}, a_5 = +\frac{a_1}{3 \cdot 5}, a_7 = -\frac{a_1}{3 \cdot 5 \cdot 7}, \dots$$

We can write solution as

$$y = a_0 \left[ 1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 - \frac{1}{2 \cdot 4 \cdot 6}x^6 + \dots + \frac{(-1)^n}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}x^{2n} + \dots \right] \\ + a_1 \left[ x - \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + \dots + \frac{(-1)^{n+1}}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n-1)}x^{2n-1} + \dots \right]$$

where  $a_0 = y(0)$  and  $a_1 = y'(0)$ .

Our Equation:  $y'' + xy' + y = 0$  has solution

$$y = a_0 \left[ 1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 - \frac{1}{2 \cdot 4 \cdot 6}x^6 + \dots + \frac{(-1)^n}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}x^{2n} + \dots \right]$$
$$+ a_1 \left[ x - \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + \dots + \frac{(-1)^{n+1}}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n-1)}x^{2n-1} + \dots \right]$$

$$y = a_0 f(x) + a_1 g(x)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n-1)}$$

**Example: Find Power Series Solution  
for  $y' = 5y + 13$  with  $y(0) = \frac{2}{5}$ .**

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + a_{n+1}x^{n+1} + \dots$$

Note first:  $a_0 = \frac{2}{5}$

Write equation as  $y' - 5y = 13$

Coefficient of  $x^n$  in  $-5y$  is  $-5a_n$

Coefficient of  $x^n$  in  $y'$  is  $(n+1)a_{n+1}$

Constant Term in  $y' - 5y$  is  $a_1 - 5a_0$  which equals 13

So  $a_1 = 13 + 5a_0 = 13 + 5\left(\frac{2}{5}\right) = 15 = 3 \times 5$

So  $(n+1)a_{n+1} - 5a_n = 0$  for  $n \geq 1$

Thus Recurrence Relation is

$$a_{n+1} = \frac{5a_n}{n+1}, \text{ for } n \geq 1$$

$n$	$a_{n+1} = \frac{5a_n}{n+1}$
1	$a_2 = \frac{1}{2} (5a_1) = \frac{1}{2} (5 \times 3 \times 5) = 3 \times \frac{5^2}{2!}$
2	$a_3 = \frac{1}{3} (5a_2) = \frac{1}{3} (5 \times 3 \times \frac{5^2}{2!}) = 3 \times \frac{5^3}{3!}$
3	$a_4 = \frac{1}{4} (5a_3) = 3 \times \frac{5^4}{4!}$
4	$a_5 = \frac{1}{5} (5a_4) = 3 \times \frac{5^5}{5!}$
...	
$n - 1$	$a_n = \frac{1}{n} (5a_{n-1}) = 3 \times \frac{5^n}{n!}$

Thus  $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots a_nx^n + \dots$

$$= \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + \dots + 3 \times \frac{(5x)^n}{n!} + \dots$$



$$y = \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + \dots + 3 \times \frac{(5x)^n}{n!} + \dots$$

$$= \frac{2}{5} + 3 \times \left( 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \dots + \frac{(5x)^n}{n!} + \dots \right)$$

$$= \frac{2}{5} + 3 - 3 + 3 \times \left( 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \dots + \frac{(5x)^n}{n!} + \dots \right)$$

$$= \frac{2}{5} - 3 + 3 \times \left( 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \dots + \frac{(5x)^n}{n!} + \dots \right)$$

$$\text{so } y = -\frac{13}{5} + 3e^{5x}$$

**Example: Solve  $y' = 5y + 13$  with  $y(0) = \frac{2}{5}$   
as First Order Linear Differential Equation.**

$$y' - 5y = 13. \text{ Integrating Factor is } e^{-5x}$$

$$(ye^{-5x})' = 13e^{-5x}$$

$$ye^{-5x} = 13\frac{-1}{5}e^{-5x} + C$$

$$y = -\frac{13}{5} + Ce^{5x}$$

$$\text{Using } y(0) = \frac{2}{5} \text{ yields } C = 3$$

## Frequently Encountered Second Order Differential Equations in Applications

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Airy	$y'' - xy = 0$
Bessel	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$
Chebyshev	$(1 - x^2)y'' - xy' + \alpha^2y = 0$
Hermite	$y'' - 2xy' + \lambda y = 0$
Laguerre	$xy'' + (1 - x)y' + \lambda y = 0$
Legendre	$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$

Equation	Areas of Application
Airy	Acoustics, Fiber Optics
Bessel	Acoustics, Electrodynamics
Chebyshev	Approximation Theory
Hermite	Quantum Mechanics
Laguerre	Approximation Theory
Legendre	Heat Flow, Electrodynamics