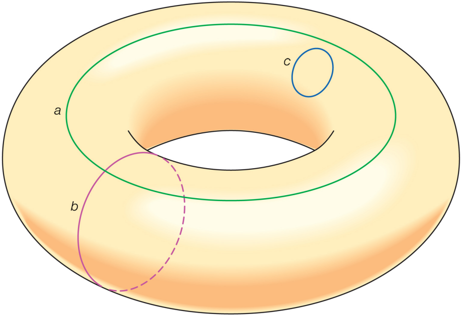
MATH 223: Multivariable Calculus



Class 35: May 13, 2022

Notes on Assignment 33

Assignment 34

Announcements

Independent Projects Due Monday

Course Response Forms: Monday
Bring Your Laptop To Class

Final Exam:
Thursday, May 19 from 2 PM to 5 PM
MATH 223 A : 311 Munroe
MATH 223 B: 317 Munroe

Good Review Questions For Final

In attempting to generalize the definition of *differentiability* of a real-valued function of a real variable introduced in beginning calculus, we have seen several different definitions advanced in this course for *derivatives* and for *differentiability* in settings involving multivariable mathematics.

What are some of the ways we have generalized the basic definition from elementary calculus?

Make sure you consider independently the cases $f : \mathcal{R}^1 \to \mathcal{R}^1, f : \mathcal{R}^1 \to \mathcal{R}^n, , f : \mathcal{R}^n \to \mathcal{R}^1,$ and $f : \mathcal{R}^n \to \mathcal{R}^m$.

In each case, show how the Chain Rule is generalized to this new setting.

Good Review Questions For Final

We have a seen a number of theorems each of which claimed to be a generalization of the *Fundamental Theorem of Calculus* first presented in beginning calculus; e.g., the theorems of Green, Gauss and Stokes together with several others.

Discuss these theorems, giving a careful statement of the hypotheses and conclusions of each one, along with a clear account of how each one might in fact be properly viewed as a generalization of the Fundamental Theorem.

Today:

Consequences of Stokes's Theorem

$$\int_S \text{curl } \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in $\mathbb{R}^3$

George Gabriel Stokes
August 13, 1819 – February 1, 1903
Stokes Biography

Interpretation of Curl

(1) The direction of curl $\mathbf{F}(\mathbf{x})$ is the axis about which $\mathbf{F}$ rotates most rapidly at $\mathbf{x}$. The length of curl $\mathbf{F}(\mathbf{x})$ is the maximum rate of rotation at $\mathbf{x}$.

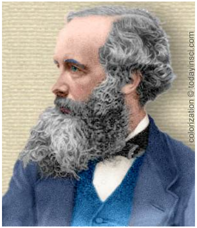
(2) **Maxwell's Equations**: curl $\mathbf{B} = \mathbf{I}$ where $\mathbf{I}$ is the vector current flow in an electrical conductor and $\mathbf{B}$ is the magnetic field which the current flow induces in the surrounding space.

Stokes's Theorem then yields **Ampere's Law**:

$$\int_S \mathbf{I} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{B} \cdot d\mathbf{x},$$

the total current flux across S is the circulation of the magnetic field around the border curve ∂S that encircles the conductor.

Definitions: A vector field $\mathbf{F}$ is **divergent-free** if div $\mathbf{F} = 0$ and $\mathbf{F}$ is **curl-free** if curl $\mathbf{F} = \mathbf{0}$.



James Clerk Maxwell (June 13, 1831 – November 5, 1879)

Maxwell Biography

Name	Equation	
	Integral form	Differential form
Faraday's law of induction	$\oint_c \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampère-Maxwell law	$\oint_c \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} + \iint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
Gauss' electric law	$\oiint_S \vec{D} \cdot d\vec{S} = \iiint_V \rho\, dV$	$\nabla \cdot \vec{D} = \rho$
Gauss' magnetic law	$\oiint_S \vec{B} \cdot d\vec{S} = 0$	$\nabla \cdot \vec{B} = 0$

Theorem: A continuously differentiable gradient field has a symmetric Jacobian matrix.

Proof: If $\mathbf{F}$ is a gradient field, then $\mathbf{F} = \nabla f$ for some real-valued function f.

Then $\mathbf{F} = (f_x, f_y)$ so the Jacobian matrix is

$$J = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

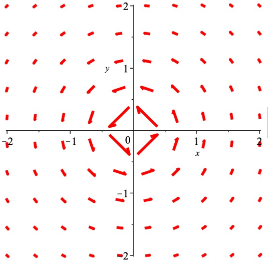
By Continuity of Mixed Partials, $f_{xy} = f_{yx}$ so J is symmetric. $\square$

Theorem: If $\mathbf{F}$ is conservative, then its Jacobian is symmetric.

Theorem: If $\mathbf{F}$ is conservative, then its Jacobian is symmetric.

The converse (Symmetric Jacobian Implies Conservative) is **FALSE** in general.

Example: Consider the vector field $\mathbf{F}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$



defined for all $(x, y) \neq (0, 0)$

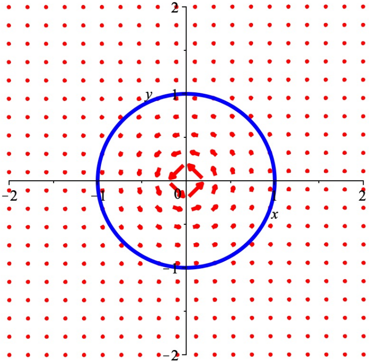
$$\text{Then Jacobian} \; = \begin{pmatrix} - & \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{y^2-x^2}{(x^2+y^2)^2} & - \end{pmatrix}$$

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Has a Symmetric Jacobian But Is Not Conservative!

If $\mathbf{F}$ were conservative, then the line integral of $\mathbf{F}$ around any closed loop would be 0.

Consider γ the unit circle as a loop running counterclockwise starting and ending at (1.0).



$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

γ: unit circle as a loop running counterclockwise starting and ending at (1.0).
We parametrize γ by $g(t) = (\cos t, \sin t), 0\pi$ so that $g'(t) = (-\sin t, \cos t)$ and

$$\mathbf{F}(g(t)) = \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right) = (-\sin t, \cos t)$$

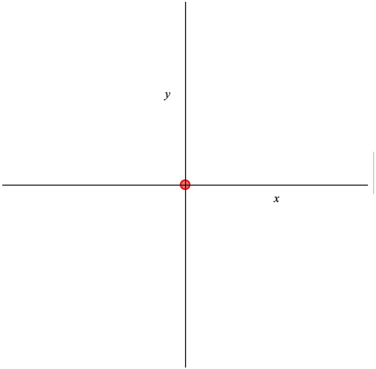
$$\mathbf{F}(g(t)) \cdot g'(t) = (-\sin t, \cos t) \cdot (-\sin t, \cos t) = \sin^2 t + \cos^2 t = 1$$

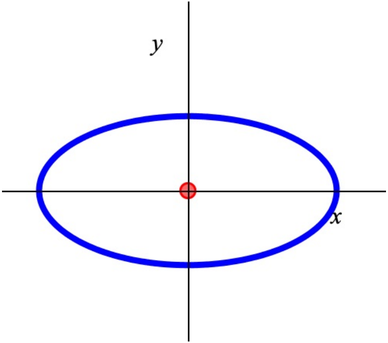
Thus $\int_\gamma \mathbf{F} = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$

What is Wrong the Vector Field

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)?$$

The Domain of the Vector Field
(Plane minus the Origin)
Is Not Simply Connected.



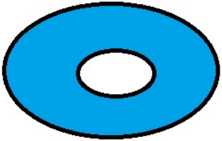


Simple Connectedness

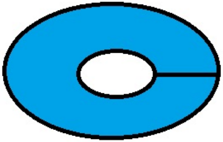
A set B is **simply connected** if every closed curve in B can be continuously contracted to a point in such a way as to stay in B during the contraction. More precisely,

Definition: An open set B is **simply connected** if every piecewise smooth closed curve lying in B is the border of some piecewise smooth orientable surface S lying in B, and with parameter domain a disk in $\mathcal{R}^2$.

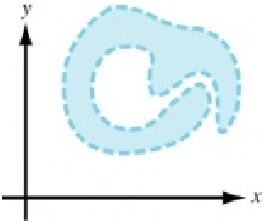
Theorem: Let $\mathbf{F}$ be a continuously differentiable vector field defined on an open set B in $\mathcal{R}^2$ or $\mathcal{R}^3$. If B is simply connected and curl F is identically zero in B, then $\mathbf{F}$ is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$



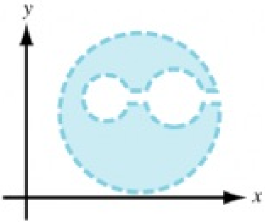
not simply connected



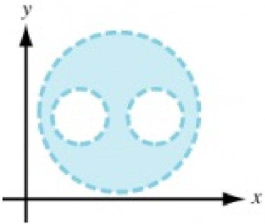
simply connected
thanks to single cut

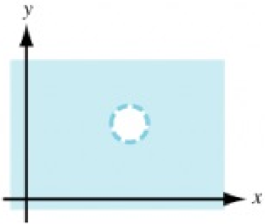


(a) A simply connected domain

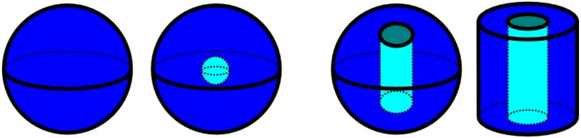


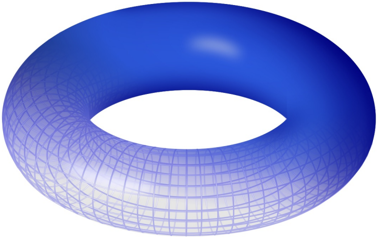
(b) A simply connected domain

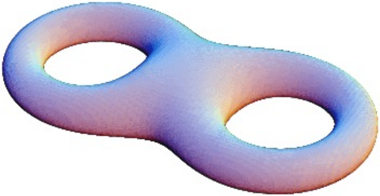




Simply connected   Non-simply connected







Theorem: Let $\mathbf{F}$ be a continuously differentiable vector field defined on an open set B in $\mathcal{R}^2$ or $\mathcal{R}^3$. If B is simply connected and curl F is identically zero in B, then $\mathbf{F}$ is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$

Proof: Let γ be a piecewise smooth closed loop in B. Because B is simply connected, there is a piecewise smooth surface S of which γ is the boundary.
By Stokes' Theorem

$$\int_\gamma \mathbf{F} = \int_S \text{curl } \mathbf{F} = \int_S \mathbf{0} = 0.$$

Thus $\mathbf{F}$ is path-independent and hence conservative. □

Theorem: Let $\mathbf{F}$ be a continuously differentiable vector field defined on an open set B in $\mathcal{R}^2$ or $\mathcal{R}^3$. If B is simply connected and curl F is identically zero in B, then $\mathbf{F}$ is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$

Theorem: If the Jacobian matrix of a continuously differentiable vector field on a simply connected set is symmetric, then the vector field is conservative.

Proof: Suppose $\mathbf{F}$ is a vector field in $\mathcal{R}^3$ with

$\mathbf{F}(\mathbf{x}) = (F(\mathbf{x}), G(\mathbf{x}), H(\mathbf{x}))$ where $\mathbf{x} = (x, y, z)$

$$Jacobian = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix} \quad \text{with} \quad \begin{matrix} F_y = G_x \\ F_z = H_x \\ G_z = H_y \end{matrix}$$

$$\text{curl } \mathbf{F} = (H_y - G_z, H_x - F_z, G_x - F_y) = (0, 0, 0) = \mathbf{0}$$