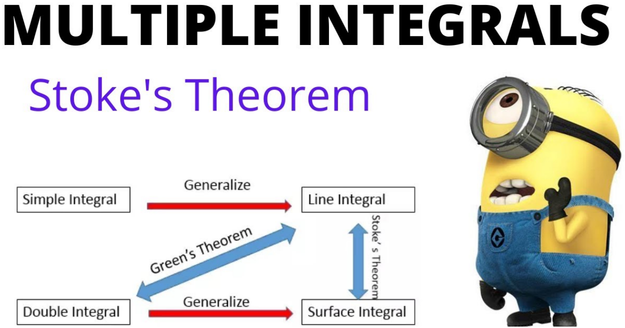
MATH 223: Multivariable Calculus



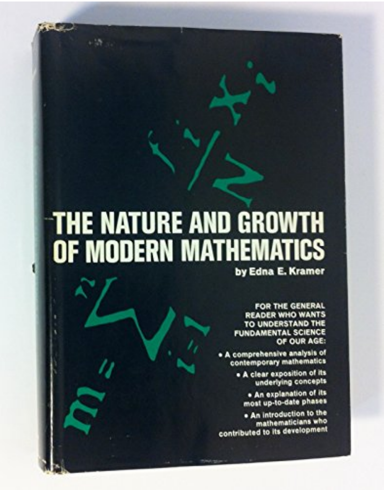
Class 34: May 11, 2022

Notes on Assignment 32

Assignment 33

Last Term's Final Exam

Mathematician of the Week



Edna Ernestine Kramer Lassar

May 11, 1902 – July 9, 1984

Announcements

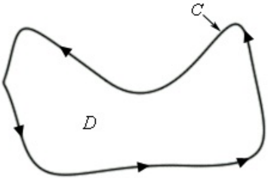
Independent Projects Due Monday

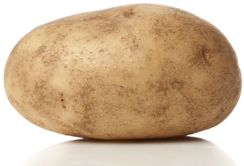
Course Response Forms: Monday
Bring Your Laptop To Class

Final Exam:
Thursday, May 19: 2 to 5 PM
MATH 223 A: Here
MATH 223 B: 317 Munroe

Gauss's Theorem aka Divergence Theorem

Planar Version: $\int_D \text{div } \mathbf{F} = \int_\gamma \mathbf{F} \cdot \mathbf{N}$





Three Dimensional Version

∂R is 2-dimensional surface surrounding 3-dimensional region R

$\int_R \text{div } \mathbf{F} = \int_{\partial R} \mathbf{F} \cdot \mathbf{N}$

Gauss's Theorem

The Setting

$\mathcal{R}$	Bounded Solid Region in $\mathbb{R}^3$
$\partial\mathcal{R}$	Finitely Many Piecewise Smooth, Closed Orientable Surfaces Oriented by Unit Normals Pointed away from $\mathcal{R}$
$\mathbf{F}$	Continuously Differentiable Vector Field in $\mathcal{R}$

The Theorem

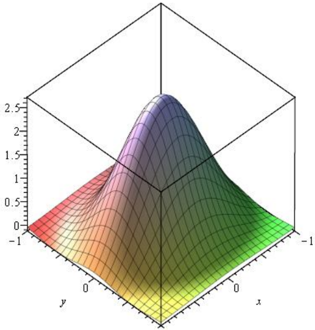
In this setting $$\int_{\mathcal{R}} \text{div } \mathbf{F} \, dV = \int_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{S}$$

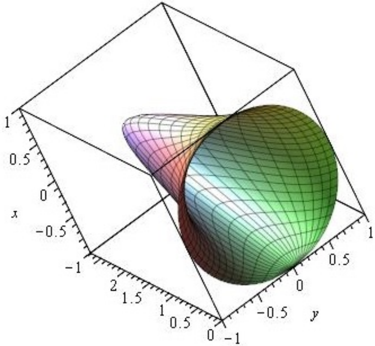
Example: $\mathbf{F} = (e^y \cos z, \sqrt{x^3+1}\sin z, x^2+y^2+3)$

div $\mathbf{F} = 0 + 0 + 0 = 0$

so $\int_R$ div $\mathbf{F} = 0$ for any region in $\mathbb{R}^3$.

Let S be graph of $z = (1-x^2-y^2)e^{1-x^2-3y^2})$ for $z \geq 0$ oriented by outward pointing unit normal vector.





Finding $\int_S \mathbf{F} \cdot d\sigma$ directly is impossible.

A Clever Way To Find $\int_S \mathbf{F} \cdot d\sigma$ indirectly.

Cap the Surface with a Disk so New Surface Bounds a 3-Dimensional Region

Form closed surface $S \cup S'$ where S' is the disk of radius 1 $(x^2 + y^2 = 1)$ in $z = 0$ plane.

Then $\int_{\partial r} \mathbf{F} = \int_{S \cup S'} \mathbf{F} = \int_S \mathbf{F} + \int_{S'} \mathbf{F}$
But by Gauss's Theorem, this integral equals 0.
Hence $\int_S \mathbf{F} = -\int_{S'} \mathbf{F}$

Now

$$\int_{S'} \mathbf{F} = -\int (--, --, x^2+y^2+3) \cdot (0,0,-1) = -\int x^2+y^2+3\,dx\,dy$$

$$= -\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2+3)\, r\, dr\, d\theta = -\frac{7}{2}\pi.$$

Thus $\int_S \mathbf{F} = \frac{7}{2}\pi$

Today:

Stokes's Theorem

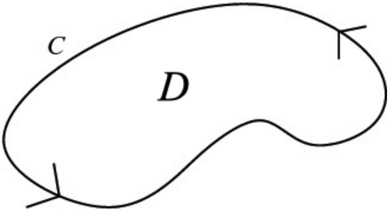
$$\int_S \text{curl } \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in $\mathbb{R}^3$

Vector Field Theorems

Plane

$\mathbf{F} : \mathcal{R}^2 \to \mathcal{R}^2$



D is 2-dimensional

$C = \partial D$ is 1-dimensional

Green – Ostrogradski	Gauss (Divergence)
$\int_D \text{curl } \mathbf{F} = \int_C \mathbf{F}$	$\int_D \text{div } \mathbf{F} = \int_C \mathbf{F} \cdot \mathbf{N}$

Positive Orientation

Setting: Let D be a plane region bounded by a curve traced out in a counterclockwise direction by some parametrization $h : \mathcal{R}^1 \to \mathcal{R}^2$ for $a \leq t \leq b$.

Let $S = g(D)$ be the image of D where $g : \mathcal{R}^2 \to \mathcal{R}^3$ so that S is a 2-dimensional surface in 3-space whose border γ corresponds to the boundary of D.

We say that γ inherits the **positive orientation** with respect to S. The composition $g(h(t))$ describes the border of S. Denote by ∂S the **positively oriented border** of S.

Vector Field in $\mathcal{R}^3$: $\mathbf{F}(\mathbf{x}) = (F(\mathbf{x}), G(\mathbf{x}), H(\mathbf{x}))$ where each of F, G, H is a real-valued function of 3 variables.

Curl $\mathbf{F}(\mathbf{x}) = (H_y(\mathbf{x}) - G_z(\mathbf{x}), F_z(\mathbf{x}) - H_x(\mathbf{x}), G_x(\mathbf{x}) - F_y(\mathbf{x}))$

Stokes's Theorem: Let S be a piece of smooth surface in $\mathcal{R}^3$, parametrized by a twice continuously differentiable function g. Assume that D, the parameter domain of g, is a finite union of simple regions bounded by a piecewise smooth curve. If $\mathbf{F}$ is a continuously differentiable vector field defined on S, then

$$\int_S \text{Curl } \mathbf{F} \cdot dS = \int_{\partial S} F \cdot d\mathbf{x}$$

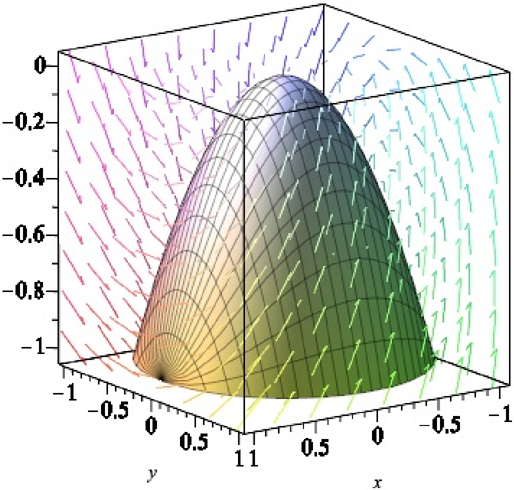
where ∂S is the positively oriented border of S.

[Note: If $\mathbf{F} = (F, G, 0)$ where F and F are independent of z, then Stokes's Theorem reduces to Green's Theorem. Thus Stokes generalizes Green.]

Example: Verify Stokes Theorem where

$$\mathbf{F}(x,y,z) = (z,x,y)$$

$$S : g(u,v) = (u,v,1-u^2-v^2), u^2+v^2 \leq 1.$$



Example: Verify Stokes Theorem where

$$\mathbf{F}(x,y,z) = (z,x,y)$$
$$S : g(u,v) = (u,v,1-u^2-v^2), u^2+v^2 \leq 1.$$

Parametrize ∂S by $(\cos t, \sin t), 0 \leq t \leq 2\pi$.

Then $g(u,v) = (\cos t, \sin t, 0)$ and $g'(u,v) = (-\sin t, \cos t, 0)$

$$\mathbf{F}\left(g(u,v)\right) = (1-u^2-v^2, u, v) = (0, \cos t, \sin t)$$

$$\mathbf{F}\left(g(u,v)\right) \cdot g'(u,v) = (0, \cos t, \sin t) \cdot (-\sin t, \cos t, 0) = \cos^2 t$$

$$\int_{\partial S} \mathbf{F} = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1+\cos 2t}{2} \, dt = \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi$$

Now $\int_S$ curl $\mathbf{F} = \int_S$ curl (z, x, y)

$$\text{curl } \mathbf{F} = det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = (1-0, -(0-1), 1-0) = (1,1,1)$$

Thus we want to integrate (1,1,1) over S.
Here $g(u,v) = (u, v, 1-u^2-v^2)$
so $g_u = (1, 0, -2u), g_v = (0, 1, -2v)$
and $g_u \times g_v = (2u, 2v, 1)$ [work it out]

$$\int_S \text{curl } \mathbf{F} = \iint_D (1,1,1) \cdot (2u, 2v, 1) \, du \, dv = \iint_D 2u + 2v + 1 \, du \, dv$$

which equals (using polar coordinates)

$$\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (2r\cos\theta + 2r\sin\theta + 1) \, r \, dr \, d\theta$$

$$\int_S \text{curl } \mathbf{F} = \iint_D (1,1,1)\cdot(2u,2v,1)\,du\,dv = \iint_D 2u+2v+1\,du\,dv$$

which equals (using polar coordinates)

$$\int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} (2r\cos\theta + 2r\sin\theta + 1)\; r\;dr\;d\theta$$

$$= \int_{r=0}^{r=1} \left[2r^2\sin\theta - 2r^2\cos\theta + r\theta\right]_{\theta=0}^{\theta=2\pi}\; dr$$

$$= \int_{r=0}^{r=1} 2\pi r\;dr = 2\pi\left[\frac{r^2}{2}\right]_{r=0}^{r=1} = \pi$$