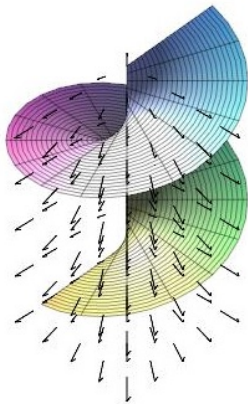


# MATH 223: Multivariable Calculus





Notes on Assignment 31  
Assignment 32  
Surface Integrals  
Final Exam of December 2021

# Announcements

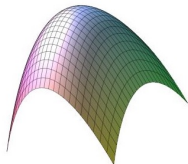
## Course Response Forms

In Class, Monday, May 16  
Bring Laptop/Smart Phone

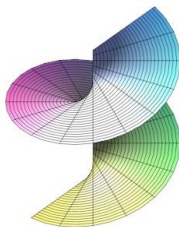
## Final Exam

Thursday, May 19  
2 – 5 PM  
Section A: Munroe 311  
Section B; Munroe 317

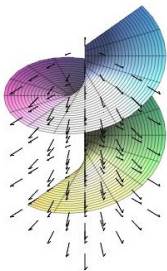
## Integrating Vector Fields Over Surfaces



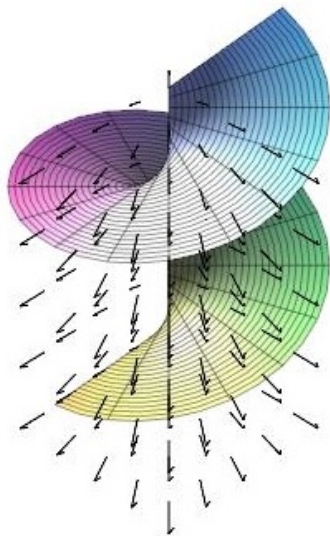
$$g(u, v) = [u, v, -2u^2 - 3v^2]$$



$$g(u, v) = [u \cos v, u \sin v, v]$$



## Integrating Vector Fields Over Surfaces



$$g(u, v) = [u \cos v, u \sin v, v]$$

Smooth Curve  $\gamma$

$$g : I \text{ in } \mathbb{R}^1 \rightarrow \mathbb{R}^n$$

$$\text{Length} = \int_I |g'(t)| dt$$

$$\text{Mass} = \int_I \mu(g(t)) |g'(t)| dt$$

Line Integral

$$\int_{\gamma} \mathbf{F} = \int_I \mathbf{F}(g(t)) \cdot g'(t) dt$$

Smooth Surface  $S$

$$g : D \text{ in } \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\text{Area } \sigma(S) = \iint_D |g_u \times g_v| dudv$$

$$\text{Mass} = \iint_D \mu d\sigma$$

Surface Integral

$$\iint_S \mathbf{F} = \iint_D \mathbf{F}(g(u, v)) \cdot (g_u \times g_v)$$

$$\iint_S \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N} d\sigma$$

$\Phi(\mathbf{F}, S) = \iint_S \mathbf{F}$  is **flux** of  $\mathbf{F}$  across  $S$ .

## Surface Integral

Let  $g$  be a function from an interval  $[t_0, t_1]$  into  $\mathbb{R}^n$  with image  $\gamma$  and  $\mu$  density at  $g(t)$ .

$$\text{Then Mass of Wire} = \int_{t_0}^{t_1} \mu(t) |g'(t)| dt$$

If  $\mu \equiv 1$ , then mass = length of curve  $\int_{t_0}^{t_1} |g'(t)| dt$

Generalize To Surfaces

Let  $D$  be region in plane and  $g : D \rightarrow \mathbb{R}^3$  with  $g(u, v) = (g_1, g_2, g_3)$  where each component function  $g_i$  is continuously differentiable.

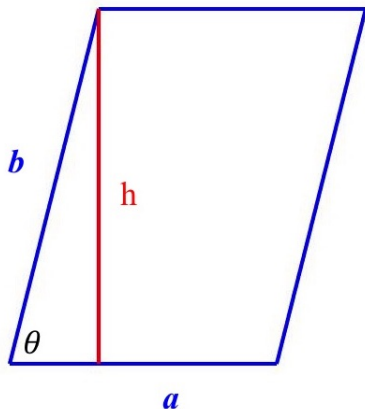
There are two natural tangent vectors:  $g_u = \frac{\partial g}{\partial u}$  and  $g_v = \frac{\partial g}{\partial v}$ ,  
These determine a tangent plane.

$S$  is a **Smooth Surface** if these two vectors are linearly independent.

Note that  $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$  is normal to the plane with

$$\left| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right| = \left| \frac{\partial g}{\partial u} \right| \left| \frac{\partial g}{\partial v} \right| \sin \theta$$

= Area of Parallelogram Spanned by the Vectors



$$\sin \theta = \frac{h}{|\mathbf{b}|} \text{ so } h = |\mathbf{b}| \sin \theta$$

$$\text{Area of Parallelogram} = (\text{Base})(\text{Height}) = |\mathbf{a}||\mathbf{b}| \sin \theta$$

$$\mathbf{a} = g_u, \mathbf{b} = g_v$$

$$|g_u \times g_v| = |g_u||g_v| \sin \theta$$



## Surface Area

$$\sigma(S) = \iint_D \left| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right| dudv = \iint_D |g_u \times g_v| dudv$$

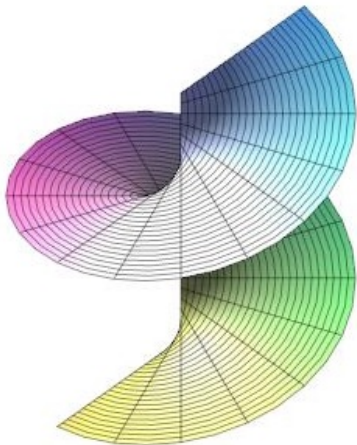
If  $\mu(g(u, v))$  is density, then mass =

$$\iint_D \mu d\sigma = \iint_D \mu(g(u, v)) |g_u \times g_v| dudv$$

Plotting Parametrized Surface in *Maple*:  
`plot3d([g1(u, v), g2(u, v), g3(u, v)], u = ..., v = ...)`

## Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$



## Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$
$$g_u = (\cos v, \sin v, 0), g_v = (-u \sin v, u \cos v, 1)$$

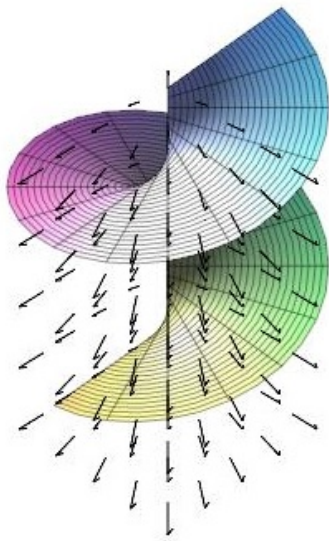
$$g_u \times g_v = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix}$$
$$= \left( \begin{vmatrix} \sin v & 0 \\ u \cos v & 1 \end{vmatrix}, - \begin{vmatrix} \cos v & 0 \\ -u \sin v & 1 \end{vmatrix}, \begin{vmatrix} \cos v & \sin v \\ -u \sin v & u \cos v \end{vmatrix} \right)$$
$$= (\sin v, -\cos v, u)$$

$$\text{Then } |g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$
$$\text{Area} = \int_{v=0}^{3\pi} \int_{u=0}^1 \sqrt{1 + u^2} \, du \, dv$$

If density is  $\mu(\mathbf{x}) = u$ , then Mass =

$$\int_{v=0}^{3\pi} \int_{u=0}^1 u(1 + u^2)^{1/2} \, du \, dv = \int_{v=0}^{3\pi} \left[ \frac{1}{3}(1 + u^2)^{3/2} \right]_0^1 \, dv$$
$$= \int_{v=0}^{3\pi} \frac{1}{3} [2^{3/2} - 1^{3/2}] \, dv = 3\pi \frac{1}{3} [2^{3/2} - 1] = \pi [2^{3/2} - 1]$$

## Integrating A Vector Field Over the Spiral Ramp



## Integrating A Vector Field Over the Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u \sin v, u \cos v, 1)$$

$$g_u \times g_v = (\sin v, -\cos v, u)$$

Suppose our vector field is  $\mathbf{F}(x, y, z) = (x^2, 0, z^2)$

$$\text{So } F(g(u, v)) = (u^2 \cos^2 v, 0, v^2)$$

The set  $D = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 3\pi\}$

$$\text{We want } \int_D F(g(u, v)) \cdot (g_u \times g_v)$$

$$\text{which equals } \int_{v=0}^{3\pi} \int_{u=0}^1 [u^2 \cos^2 v \sin v + uv^2] du dv$$

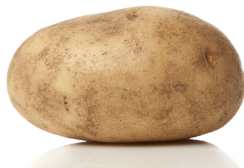
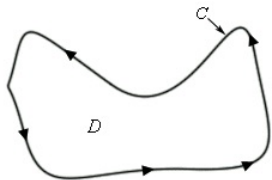
$$= \int_{v=0}^{3\pi} \left[ \frac{u^3}{3} \cos^2 v \sin v + \frac{u^2}{2} v^2 \Big|_{u=0}^1 \right] dv =$$

$$\int_{v=0}^{3\pi} \left[ \frac{1}{3} \cos^2 v \sin v + \frac{1}{2} v^2 \right] dv$$

$$= \left[ \frac{-\cos^3 v}{9} + \frac{v^3}{6} \right]_{v=0}^{3\pi} = \frac{1}{9} + \frac{3^3 \pi^3}{6} - \frac{-1}{9} = \frac{2}{9} + \frac{9}{2} \pi^3$$

## Gauss's Theorem aka Divergence Theorem

$$\text{Planar Version: } \int_D \text{div } \mathbf{F} = \int_\gamma \mathbf{F} \cdot \mathbf{N}$$



Three Dimensional Version

$\partial R$  is 2-dimensional surface surrounding 3-dimensional region  $R$

$$\int_R \text{div } \mathbf{F} = \int_{\partial R} \mathbf{F} \cdot \mathbf{N}$$

## Gauss's Theorem

### The Setting

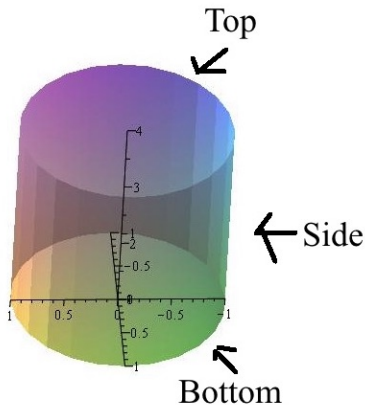
- $\mathcal{R}$  Bounded Solid Region in  $\mathbb{R}^3$
- $\partial\mathcal{R}$  Finitely Many Piecewise Smooth, Closed Orientable Surfaces  
Oriented by Unit Normals Pointed away from  $\mathcal{R}$
- $\mathbf{F}$  Continuously Differentiable Vector Field in  $\mathcal{R}$

### The Theorem

In this setting 
$$\int_{\mathcal{R}} \operatorname{div} \mathbf{F} dV = \int_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{S}$$

Example Verify Gauss's Theorem where  $\mathcal{R}$  is solid cylinder of radius  $a$  and height  $b$  with the  $z$ -axis as the axis of the cylinder and

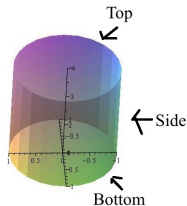
$$\mathbf{F} = (x, y, z)$$



$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{Bottom} \mathbf{F} \cdot d\mathbf{S} + \int_{Top} \mathbf{F} \cdot d\mathbf{S} + \int_{Side} \mathbf{F} \cdot d\mathbf{S}$$



Cylinder of Radius  $a$  and height  $b$



$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{Bottom} \mathbf{F} \cdot d\mathbf{S} + \int_{Top} \mathbf{F} \cdot d\mathbf{S} + \int_{Side} \mathbf{F} \cdot d\mathbf{S}$$

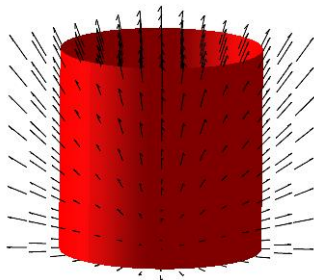
For  $\int_{Bottom} \mathbf{F} \cdot d\mathbf{S}$ , unit normal is  $(0,0,-1)$

Then  $(x, y, z) \cdot (0, 0, -1) = -z$  but  $z = 0$  so  $\int_{Bottom} \mathbf{F} \cdot d\mathbf{S} = 0$

For  $\int_{Top} \mathbf{F} \cdot d\mathbf{S}$ , unit normal is  $(0,0,1)$

Then  $(x, y, z) \cdot (0, 0, +1) = z$  but  $z = b$  so  $\int_{Top} \mathbf{F} \cdot d\mathbf{S}$   
is  $b \times \text{area of top} = b\pi a^2$

Finally,  $\int_{Side} \mathbf{F} \cdot d\mathbf{S}$



**Vector Field**  $\mathbf{F} = (x, y, z)$

**Surface:**  $x^2 + y^2 = a^2, 0 \leq z \leq b$

$g(u, v) = (a \cos u, a \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq b$

Finally,  $\int_{Side} \mathbf{F} \cdot dS$

$$g(u, v) = (a \cos u, a \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq b$$

$$g_u = (-a \sin u, a \cos u, 0), \quad g_v = (0, 0, 1)$$

$$g_u \times g_v = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\text{expanding along bottom row}) (a \cos u, a \sin u, 0)$$

$$\text{Thus } |g_u \times g_v| = \sqrt{a^2 \cos^2 u + a^2 \sin^2 u + 0^2} = a$$

$$\text{Also } F(g(u, v)) = (a \cos u, a \sin u, v) \text{ so } F(g(u, v)) \cdot (g_u \times g_v) = a^2 \cos^2 u + a^2 \sin^2 u + 0 = a^2.$$

$$\text{so } \int_{Side} \mathbf{F} \cdot dS = \int_{v=0}^b \int_{u=0}^{2\pi} a^2 du dv = 2\pi a^2 b$$

$$\text{Putting it altogether: } \int_S \mathbf{F} \cdot dS$$

$$= \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS = 0 + \pi a^2 b + 2\pi a^2 b = 3\pi a^2 b$$

On The Other Hand, we compute  $\int_R \operatorname{div} \mathbf{F}$

$$\mathbf{F} = (x, y, z)$$

$$\operatorname{div} \mathbf{F} = 1 + 1 + 1 = 3$$

The solid  $R$  is more easily described in polar coordinates

$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq a \quad 0 \leq z \leq b.$$

$$\int_R \operatorname{div} \mathbf{F} = \int_{\theta=0}^{2\pi} \int_{z=0}^b \int_{r=0}^a \operatorname{div} \mathbf{F} r dr dz d\theta = \int_{\theta=0}^{2\pi} \int_{z=0}^b \int_{r=0}^a 3r dr dz d\theta$$

$$\int_{\theta=0}^{2\pi} \int_{z=0}^b 3 \frac{r^2}{2} \Big|_{r=0}^a dz d\theta = \int_{\theta=0}^{2\pi} \int_{z=0}^b \frac{3}{2} a^2 dz d\theta = \int_{\theta=0}^{2\pi} \frac{3}{2} a^2 b d\theta = 2\pi \frac{3}{2} a^2 b$$

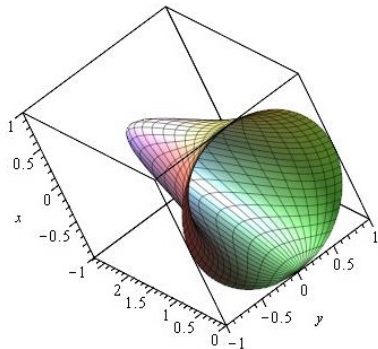
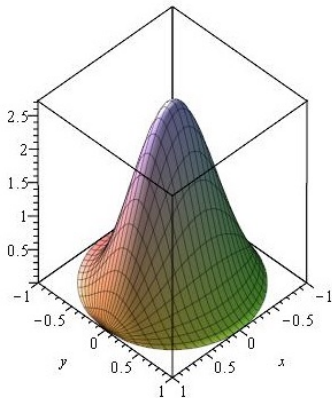
$$= 3a^2 b \pi$$

Example:  $\mathbf{F} = (e^y \cos z, \sqrt{x^3 + 1} \sin z, x^2 + y^2 + 3)$

$$\operatorname{div} \mathbf{F} = 0 + 0 + 0 = 0$$

so  $\int_R \operatorname{div} \mathbf{F} = 0$  for any region in  $\mathbb{R}^3$ .

Let  $S$  be graph of  $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$  for  $z \geq 0$   
oriented by outward pointing unit normal vector.



Finding  $\int_S \mathbf{F} \cdot d\sigma$  directly is impossible.

## A Clever Way To Find $\int_S \mathbf{F} \cdot d\sigma$ indirectly.

Cap the Surface with a Disk so New Surface Bounds a 3-Dimensional Region

Form closed surface  $S \cup S'$  where  $S'$  is the disk of radius 1 ( $x^2 + y^2 = 1$ ) in  $z = 0$  plane.

$$\text{Then } \int_{\partial V} \mathbf{F} = \int_{S \cup S'} \mathbf{F} = \int_S \mathbf{F} + \int_{S'} \mathbf{F}$$

But by Gauss's Theorem, this integral equals 0.

$$\text{Hence } \int_S \mathbf{F} = - \int_{S'} \mathbf{F}$$

Now

$$\begin{aligned} \int_{S'} \mathbf{F} &= - \int (-, -, x^2 + y^2 + 3) \cdot (0, 0, -1) = \int x^2 + y^2 + 3 \, dx \, dy \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^1 (r^2 + 3) r \, dr \, d\theta = \frac{7}{2}\pi \end{aligned}$$

Next Time:

## Stokes's Theorem

$$\int_S \operatorname{curl} \mathbf{F} = \int_{\partial S} \mathbf{F}$$

$S$  is a Surface in  $\mathbb{R}^3$