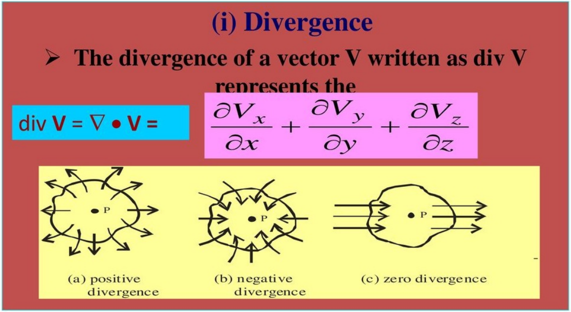
MATH 223: Multivariable Calculus

Divergence, Curl and Gradient Operations



Class 29: November 19, 2021

Notes on Assignment 28
Assignment 29
Divergence and Curl

Announcements

Exam 3: Wednesday, December 1
One Page of Notes

Today

Begin Chapter 8: Vector Field Theory

Divergence and Curl: Measures of Rates of Change of Vector Fields

Divergence of a Vector Field

Definition div $\mathbf{F}$ = trace of $\mathbf{F}'$ of the Jacobi Matrix

Example **F**: $\mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x,y) = (2x - y, x - 3y)$

$$\mathbf{F} = \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} \text{ implies div } \mathbf{F} = 2 - 3 = -1$$

Example: **F**: $\mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x,y,z) = (xy, yz, zx)$

$$\mathbf{F'} = \begin{pmatrix} y & -- & -- \\ -- & z & -- \\ -- & -- & x \end{pmatrix} \text{ implies div } \mathbf{F} = y + z + x$$

Example: **F**: $\mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x, y, z) = (yz, xz, xy)$

Alternate Notation: $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$$\mathbf{F'} = \begin{pmatrix} 0 & z & y \\ z & 0 & y \\ y & z & 0 \end{pmatrix} \text{ implies div } \mathbf{F} = 0$$

$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$$

$$\mathbf{F'} = \begin{matrix} F1 \\ F2 \\ F3 \end{matrix} \begin{pmatrix} 0 & z & y \\ z & 0 & y \\ y & z & 0 \end{pmatrix} \text{ implies div } \mathbf{F} = 0$$

In general, **div F is a real -valued function of n variables**.

Notes

1. Gauss's Theorem: $\int_R \text{ div } \mathbf{F}\, dV = \int_{\partial R} \mathbf{F} \cdot d\mathbf{S}$

2. div $\mathbf{F}$ gives expansion rate of fluid at point $\mathbf{x}$
   div $\mathbf{F} > 0$ means fluid is expanding, getting less dense
   div $\mathbf{F} < 0$ means fluid is contracting, becomes more dense

3. Alternate Notation; $\mathbf{F} = (F_1, F_2, F_3), \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
   Then div $\mathbf{F} = \mathbf{F} \cdot \nabla$

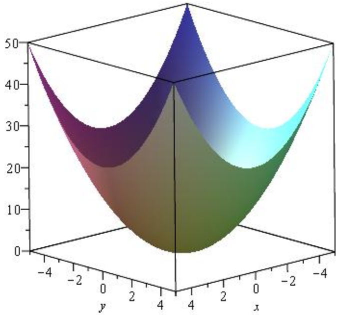
Example

$$\mathbf{F}(x,y,z) = (xy^2 + z\ln(1+y^2), \sin(xz) - zy, x^2z + \arctan y + e^{x^2})$$

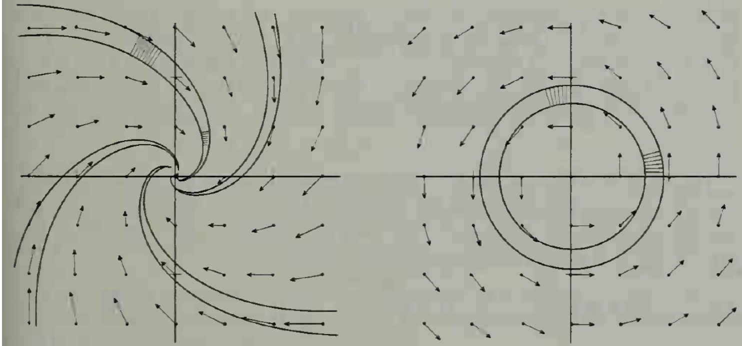
$$\text{div } \mathbf{F} = y^2 - z + x^2$$

$$\text{so div } \mathbf{F} > 0 \text{ if } x^2 + y^2 > z$$

$z = x^2 + y^2$ is equation of elliptic paraboloid.



Divergence is positive on the outside, negative on the inside.



$\mathbf{F}(x, y) = \frac{1}{8}(-x + y)\mathbf{i} + \frac{1}{8}(-x - y)\mathbf{j}$
Area decreased: $\operatorname{div}\mathbf{F}(x, y) = -\frac{1}{4}$

(a)

$\mathbf{G}(x, y) = -\frac{1}{4}(y/\sqrt{x^2 + y^2})\mathbf{i} + \frac{1}{4}(x/\sqrt{x^2 + y^2})\mathbf{j}$
Area preserved: $\operatorname{div}\mathbf{G}(x, y) = 0$

(b)

Figure 8.10

Curl of a Vector Field

Curl measures local tendency of a vector field and its flow lines to circulate around some axis.

The curl of a vector field is itself a vector field.

Setting; $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is our vector field

$\mathbf{F} = (F_1, F_2, F_3)$ so $\mathbf{F}(x, yz) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$

Formal Definition: curl $\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$

Mnemonic Device:

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}. \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Expand along first row:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \mathbf{k}$$

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Example: $\mathbf{F}(x, y, z) = (xyz, y - 3z, 2y)$

$$\text{curl } \mathbf{F} = ((2y)_y - (-3z)_z, (xyz)_z - (2y)_x, (y - 3z)_x - (xyz)_y)$$

$$= (2 - (-3), xy - 0, 0 - xz) = (5, xy, -xz)$$

Scalar Curl for Vector Fields in Plane

F $= (F, G, 0)$ where$F(x, y)$ and $G(x, y)$are functions only of x and y.

Then curl **F** $= (0, 0, G_x - F_y)$

Note: Curl and Conservative Vector Field

Suppose **F** $= (F, G, 0)$ is gradient field with **F** $= \nabla f$.

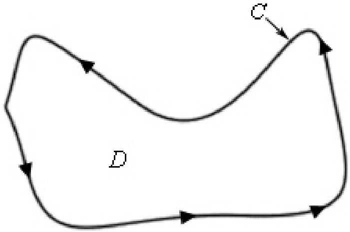
Then $F = f_x$ and $G = f_y$

In this case, Curl **F** $= (0, 0, f_{yx} - f_{xy}) = (0, 0, 0)$

by Clairault's Theorem on Equality of Mixed Partials.

Green's Theorem in the Plane

$$\iint_D \text{curl}\,\mathbf{F} = \int_\gamma \mathbf{F}$$



D is bounded plane region.

$C = \gamma$ is piecewise smooth boundary of D

F and G are continuously differentiable functions defined on D

Then

$$\int\int (G_x - F_y)dxdy = \int_\gamma Fdx + Gy$$

where γ is parametrized so it is traced once with D on the left.

Application of Green's Theorem in the Plane

$$\iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$$

Example $\mathbf{F}(x, y) = (0, x)$ implies $curl\mathbf{F} = 1 - 0 = 1$

Hence $\iint_D \text{curl } \mathbf{F} = \iint_D 1 =$ area of D

Green's Theorem enables us to find the area of a planar region if we can develop a parametrization of its boundary.

Example Consider the unit disk D of radius r centered at origin.

Let $g(t) = (r\cos t, r\sin t), 0 \leq t \leq 2\pi$

So $g'(t) = (r\sin t, r\cos t)$

and $\mathbf{F}(g(t)) = (0, r\cos t)$

Then $\mathbf{F}(g(t)) \cdot g'(t) = r^2\cos^2 t\, dt$

Thus area of $D = \iint_D 1 = \iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F} = \int_0^{2\pi} r^2\cos^2 t\, dt$

$\int_0^{2\pi} r^2\cos^2 t\, dt = r^2\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt = \frac{r^2}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi} \pi r^2$

Using Green's Theorem

(1) Compute $\iint_D \operatorname{curl} \mathbf{F}$ by using $\int_\gamma \mathbf{F}$

(2) Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \operatorname{curl} \mathbf{F}$

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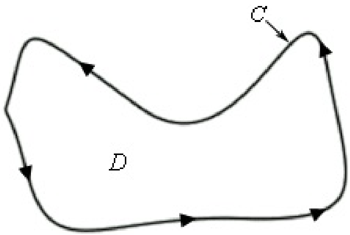

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Using Green's Theorem

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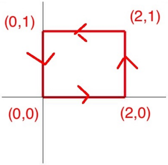
(2) Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \text{curl}\,\mathbf{F}$

Example

Find

$$\int_\gamma (1+10xy+y^2)\,dx+(6xy+5x^2)\,dy = \int_\gamma (1+10xy+y^2, 6xy+5x^2)$$

where γ is boundary of the rectangle with vertices (0,0), (2,0), (2,1), and (0,1).



Note: Direct Computation requires 4 integrals.

$$F(x,y) = 1 + 10xy + y^2. \quad G(x,y) = 6xy + 5x^2$$

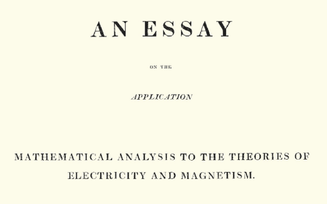
$$F_y = 10x + 2y \qquad . \; G_x = 6y + 10x$$

$$G_x - F_y = 6y + 10x - 10x - 2y = 4t$$

$$\int_\gamma \mathbf{F} = \iint_D \text{curl}\,\mathbf{F} = \int_0^2 \int_0^1 4y\,dy\,dx = \int_0^2 \left[2y^2\right]_0^1 = \int_0^2 2dx = 4$$

George Green
1793 – 1841

Mikhail Ostrogradsky
1801 – 1861



AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF
ELECTRICITY AND MAGNETISM.