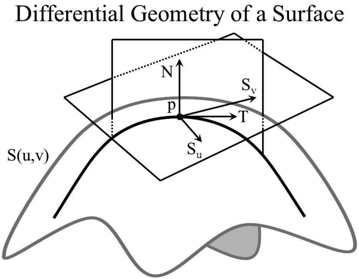
MATH 223: Multivariable Calculus



Class 29: April 29, 2022

Further Note on Assignment 26

Exercise 7: Find a potential function for $\mathbf{F}(x, y) = (2xe^y - \sin x \sin y, x^2 e^y + \cos x \cos y)$. "

We want a function f such that $f_x(x, y) = 2xe^y - \sin x \sin y$ and $f_y(x, y) = x^2 e^y + \cos x \cos y$.

If we integrate the first component $2xe^y - \sin x \sin y$ with respect to x, we obtain a function $f(x, y) = x^2 e^y + \cos x \sin y + H(y)$ whose derivative with respect to y is the second component of $\mathbf{F}$ if $H'(y) = 0$ so choose $H(y) = 0$.

Handouts

Notes on Assignment 27
Assignment 28
Normal Vectors and Curvature

"Sit and stay were no problem but she's hit a wall with multivariable calculus."

Exam 3: Monday Night at 7 PM Axinn 229

You May Bring One Sheet (Two-Sided) of Notes

Announcements

Chapter 7: Integrals and Derivatives on Curves

Today: Normal Vectors and Curvature

Monday: Flow Lines, Divergence and Curl

Normal Vectors and Curvature

Goal: Derive a Measure of Shape of a Curve.

How "Curvy" is a Curve?

Setting: Curve γ lies in $\mathbb{R}^2$ or $\mathbb{R}^3$

Parametrization $\mathbf{g}$ whose image is γ.

Some texts use $\mathbf{r}$ or $\mathbf{x} = \mathbf{x}(t)$ for the parametrization

Arc Length traversed by time t is denoted $s(t)$ and is a scalar quantity with

$$s(t) = \int |\mathbf{g}'(t)|\, dt$$

Arc Length is Integral of Speed

Speed is Derivative of Arc Length:

$$s'(t) = |\mathbf{g}'(t)|$$

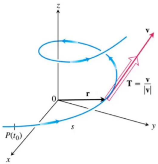
so we will have $\mathbf{g}'(t) = s't)\mathbf{T}(t)$

where **T** is unit tangent vector.

Unit Tangent Vector

The **unit tangent vector** gets its own notation:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{|\vec{\mathbf{r}}\,'(t)|} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$



$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$$

<u>Example</u> $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2+b^2}$

$$\mathbf{T}(t) = \frac{g'(t)}{|g'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

Principal Normal Vector

Start With Observation: $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$

Now differentiate both sides with respect to t:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 2\mathbf{T} \cdot \mathbf{T}' = 0$$

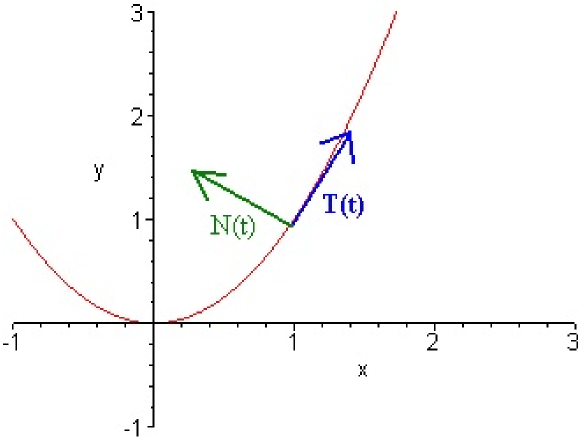
So $\mathbf{T} \cdot \mathbf{T}' = 0$

The vectors $\mathbf{T}$ and $\mathbf{T}'$ are Orthogonal

The Principal Normal Vector

$$\eta(t) = \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

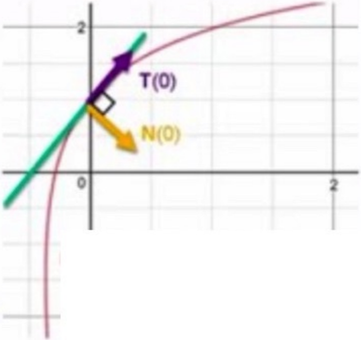
Sometimes written as $\mathbf{N} = \frac{\mathbf{T'}}{|\mathbf{T'}|}$ or $\mathbf{n} = \frac{\mathbf{t}}{|\mathbf{t}|}$


3
2
1
-1
y
-1
1
2
3
x
N(t)
T(t)

Principal Unit Normal Vector

- $0 = 1' = (T \cdot T)'$
  $= T' \cdot T + T \cdot T'$
  $= 2T \cdot T'$

- $$N(t) = \frac{T'(t)}{\|T'(t)\|}$$



Principal Normal

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

Example $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2 + b^2}$

$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

$$\mathbf{N} = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}} \times \frac{\sqrt{a^2+b^2}}{a} = \frac{(-a\cos t, -a\sin t, 0)}{a}$$

$$\mathbf{N} = (-\cos t, -\sin t, 0)$$

$$\mathbf{N} \cdot \mathbf{T} = \frac{a\sin t\cos t - a\sin t\cos t + 0}{\sqrt{a^2+b^2}} = 0.$$

Example: Parabola in the Plane

$$\mathbf{g}(t) = (t, t^2)$$

$$\mathbf{g}'(t) = (1, 2t)$$

$$|\mathbf{g}'(t)| = \sqrt{1 + 4t^2}$$

$$\mathbf{T} = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(1,2t)}{\sqrt{1+4t^2}} = \left((1 + 4t^2)^{-1/2}, 2t(1 + 4t^2)^{-1/2}\right)$$

Differentiating with respect to t and simplifying, we get

$$\mathbf{T}' = \left(\frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right)$$

After some algebra, $|\mathbf{T}'| = \frac{2}{1+4t^2}$

$$\mathbf{N} = \left(\frac{-2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}},\right)$$

Check that $\mathbf{N} \cdot \mathbf{T} = 0$

Curvature

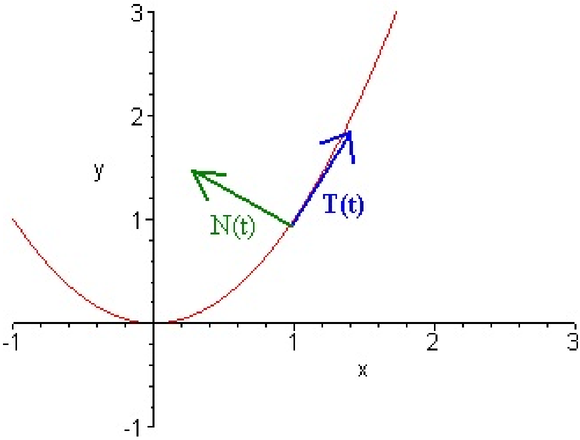
Recall $s'(t) = |\mathbf{g}'(t)|$ or, more compactly, $s' = |\mathbf{g}'|$
and $\mathbf{T} = \frac{\mathbf{g}'}{|\mathbf{g}'|} = \frac{\mathbf{g}'}{s'}$ we have $\mathbf{g}' = s'\mathbf{T}$.
Differentiate with respect to t:

$$\mathbf{g}'' = \mathbf{g}'' = (s'\mathbf{T})' = s''\mathbf{T} + s'\mathbf{T}'$$

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'\mathbf{T}'$
acceleration vector		component in direction of $\mathbf{T}$		component in direction of $\mathbf{T}'$

Replace $\mathbf{T}'$ by $|\mathbf{T}'|\mathbf{N}$:

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'|\mathbf{T}'|\mathbf{N}$
acceleration vector		tangential acceleration		centripetal acceleration


3
2
1
-1
-1
1
2
3
y
x
N(t)
T(t)

Curvature

$$\underset{\text{acceleration vector}}{\mathbf{g}''} = \underset{\text{tangential acceleration}}{s''\mathbf{T}} + \underset{\text{centripetal acceleration}}{s'|\mathbf{T}'|\mathbf{N}}$$

Curvature is a measure of the bend

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right|$$

<u>Theorem</u>: $\kappa = \frac{|\mathbf{T}'|}{s'} = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}$.

Proof: $\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{\mathbf{T}'}{s'}$

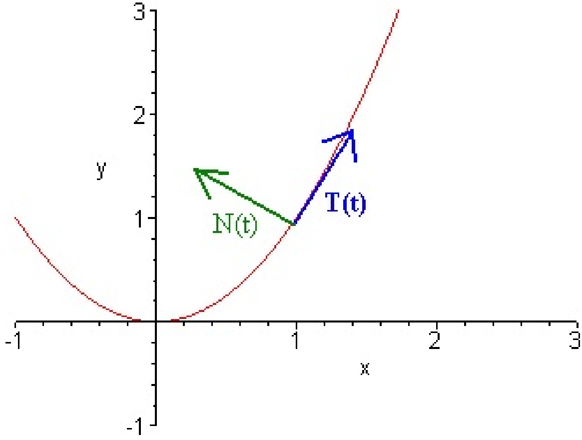
$$\boxed{\kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}}$$

$$\text{Curvature: } \kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}$$

Example Our Parabola $\mathbf{g}(t) = (t, t^2)$

We found $|\mathbf{T}'| = \frac{2}{1+4t^2}$ and $|\mathbf{g}'(t)| = \sqrt{1+4t^2}$

Thus Curvature $= \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|} = \frac{2}{1+4t^2} \times \frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}$

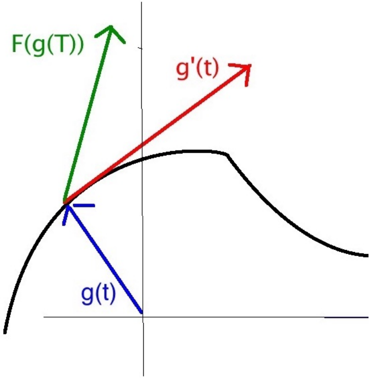


Flow Lines

Suppose γ is a curve in $\mathbb{R}^n$ which has a parametrization g.
At each point on the curve, we can associate two vectors:

Tangent Vector: $\mathbf{g'}(t)$

Vector Field: $\mathbf{F}(\mathbf{g}(t))$



If the two vectors coincide, then γ is called a **flow line** for **F**.

Hard Problem: Given $\mathbf{F}$, find flow lines
(Central Question in Differential Equations)

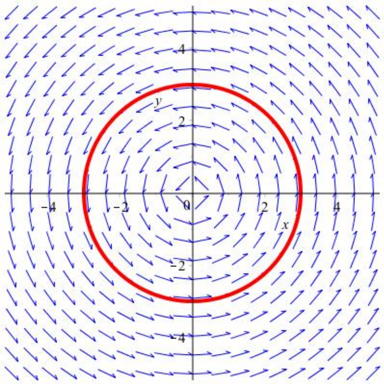
Easy Problem: Given $\mathbf{g}$ and $\mathbf{F}$, check if γ is a flow line for $\mathbf{F}$.

Example: $\mathbf{g}(t) = (3\cos\frac{t}{12}, 3\sin\frac{t}{12})$

Then $\mathbf{g'}(t) = (-\frac{1}{4}\sin\frac{t}{12}, \frac{1}{4}\cos\frac{t}{12})$

Suppose $\mathbf{F}(x,y) = \left(\frac{-y}{4\sqrt{x^2+y^2}}, \frac{x}{4\sqrt{x^2+y^2}}\right)$

Then $\mathbf{F}(x,y) = \left(\frac{-3\sin\frac{t}{12}}{4\times 3}, \frac{3\cos\frac{t}{12}}{4\times 3}\right) = \mathbf{g'}(t)$



Flow Lines and Differential Equations

Star with a system of differential equations

$$\frac{dx}{dt} = (2-y)(x-y) = f(x,y)$$
$$\frac{dy}{dt} = (1+x)(x+y) = g(x,y)$$

Can write as a single equation:
$\frac{dy}{dx} = \frac{(1+x)(x-y)}{(2-y)(x-y)} = \frac{g(x,y)}{f(x,y)}$
Observe:

1. Solution of the equation is a curve in the (x,y)-plane
2. As time goes forward, point moves along the curve in accordance to the equation
3. $\mathbf{F}(x,y) = (f(x,y), g(x,y))$ is a vector field.
4. At each point on curve, direction of motion is given by the vector field
5. The vector field is tangent to the curve
6. The curve is tangent to the vector field

<u>Definition</u>:A **flow line** of a vector field **F** is a differentiable function **g** such that the velocity vector **g'** at each point coincides with the field vector **F(g)**.