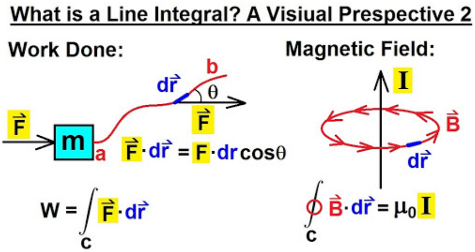
MATH 223: Multivariable Calculus



Class 27: April 28, 2022

Handouts

Notes on Assignment 25

Assignment 26

Integrals and Derivatives on Curves

Sample Exam 3

Next Monday Evening

Independent Project:

Opportunity to Study Topic in Depth

10 – 12 Hours of Work

5 - 8 Pages

Due: Monday, May 16
(In Class)

Announcements

Chapter 7: Integrals and Derivatives on Curves

Today: Line integral

Next Topics:
Weighted Curves and Arc Length
Surfaces of Revolution
Normal Vectors and Curvature

Integrals So Far

Real Valued Functions: $f : \mathbb{R}^n \to \mathbb{R}^1$

Iterated Integral

Multiple Integral

Vector Valued Functions

(A): $f := \mathbb{R}^1 \to \mathbb{R}^n$

$$f(\vec{t}) = (f_1(t), f_2(t), ...f_n(t))$$

so $\int_a^b f(\vec{t})\, dt = (\int_a^b f_1(t), \int_a^b f_2(t), ..., \int_a^b f_n(t)$

(B):VECTOR FIELDS $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n$

$$\mathbf{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), .., F_n(\vec{x}))$$

What is Meaning of $\int_{\mathcal{D}} \mathbf{F}$?

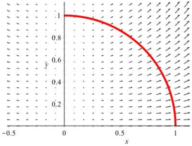
Today: $\mathcal{D}$ is a one-dimensional set in $\mathbb{R}^n$
$\mathcal{D}$ is a curve defined by a function $g : \mathbb{R}^1 \to \mathbb{R}^n$ on an interval $a \leq t \leq b$
We denote the **image** of g by γ
Definition The **Line Integral** of $\mathbf{F}$ over γ is

$$\int_\gamma \mathbf{F} \cdot d\vec{x} = \int_a^b \mathbf{F}(g(t)) \cdot g'(t)\, dt$$

Example

Curve: $g(t) = (\cos t, \sin t), 0 \leq t \leq \frac{\pi}{2}$

Vector Field: $\mathbf{F}(x, y) = (x, yx^2)$



Then $\mathbf{F}(g(t)) = \mathbf{F}(\cos t, \sin t) = (\cos t, \sin t \cos^2 t)$ and

$$g'(t) = (-\sin t, \cos t)$$

Hence $\mathbf{F}(g(t)) \cdot g'(t) = (\cos t, \sin t \cos^2 t) \cdot (-\sin t, \cos t) = -\sin t \cos t + \sin t \cos^2 t \cos t = -\sin t \cos t + \sin t \cos^3 t$

so $\int_\gamma \mathbf{F} = \int_0^{\pi/2} (-\sin t \cos t + \sin t \cos^3 t)\, dt$

$$= \left[\frac{\cos^2 t}{2} - \frac{\cos^4 t}{4} \right]_0^{\pi/2} = 0 - 0 - \frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$

Alternative Notation for $n = 2$

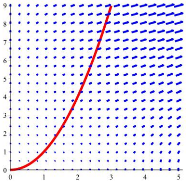
$$g(T) = (g_1(t), g_2(t)) = (x(t), y(t))$$

$$\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$$

$$\int_\gamma \mathbf{F} \cdot d\vec{x} = \int_\gamma (F_1 dx + F_2 dy)$$

In our example, $\int_\gamma (x dx + y x^2 dy)$

Example: Find $\int_\gamma \mathbf{F}$ where $\mathbf{F}(x,y) = (2xy, x^2+2y)$ and γ is the graph of $y = x^2$ from $x = 0$ to $x = 3$.



Solution: First, find a parametrization of γ.
Here $g(t) = (t, t^2), 0 \leq t \leq 3$ will work.
Then $g'(t) = (1, 2t)$ and
$\mathbf{F}(g(t)) = F(t, t^2) = (2t^3, t^2 + 2t^2) = (2t^3, 3t^2)$
so $\mathbf{F}(g(t)) \cdot g'(t) = 2t^3 + 6t^3 = 8t^3$

and $\int_\gamma \mathbf{F} = \int_0^3 8t^3\,dt = 2t^4\Big|_0^3 = 162.$

What If We Used A Different Parametrization?

$$\mathbf{F}(x,y) = (2xy, x^2 + 2y)$$

Example: Let $h(t) = (\sqrt{t}, t)$ on $0 \leq t \leq 9$

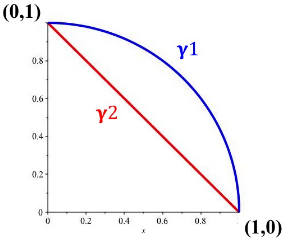
Then $h'(t) = (\frac{1}{2\sqrt{t}}, 1)$

Here $\mathbf{F}(h(t)) = \mathbf{F}((\sqrt{t}, t)) = (2t\sqrt{t}, t + 2t) = (2t^{3/2}, 3t)$

$$\int_\gamma \mathbf{F} = \int_0^9 [W] \; dt = \int_0^9 4t \, dt = 2t^2 \Big|_0^9 = 162$$

Theorem The value of the line integral $\int_\gamma \mathbf{F}$ is independent of the parametrization of γ but in general is dependent on the curve itself.

Proof: Use Change of Variable Formula; see text.



For some vector fields, the line integral $\int_\gamma \mathbf{F}$ depends only on the **endpoints** of the curve.

Theorem (**The Fundamental Theorem of Calculus for Line Integrals**. Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuously differentiable and let $\mathbf{F} = \nabla f$ and suppose $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous curve with endpoints $\vec{a}$ and $\vec{b}$.

Then $\int_\gamma \mathbf{F} = \int_\gamma \nabla f = f(\vec{b}) - f(\vec{a})$.

<u>Theorem</u> (**The Fundamental Theorem of Calculus for Line Integrals**. Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuously differentiable and let $\mathbf{F} = \nabla f$ and suppose $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous curve with endpoints $\vec{a}$ and $\vec{b}$.

Then $\int_\gamma \mathbf{F} = \int_\gamma \nabla f = f(\vec{b}) - f(\vec{a})$.

<u>Proof</u>: Let g be any parametrization of γ. with $g(0) = \vec{a}$ and $g(1) = \vec{b}$.

Thus $\mathbb{R}^1 \to g \to \mathbb{R}^n \to f \to \mathbb{R}^1$

Use Our Old Friend **The Chain Rule**;

$$[f(g(t)]' = f'(g(t)) \cdot g'(t)$$
$$= \nabla f(g(t)) \cdot g'(t)$$
$$= \mathbf{F}(g(t)) \cdot g'(t)$$

Hence $\int_\gamma \mathbf{F} = \int_0^1 \mathbf{F}(g(t)) \cdot g'(t)\, dt$

$$= \int_0^1 [f(g(t)]'\, dt$$
$$= [f(g(t)]\Big|_0^1$$
$$= f(g(1)) - f(g(0)) = f(\vec{b}) - f(\vec{a})$$

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Example: $f(x, y) = x^2 y + y^2$

so $\mathbf{F} = \nabla f = (2xy, 2y)$

let γ be any curve from (0,0) to (4,2)

Then $\int_\gamma \mathbf{F} = f(4, 2) - f(0, 0) = 4^2 \times 2 + 2^2 - (0 + 0) = 36$

If **F** $= \nabla f$ for some f, then we call **F** a **Conservative Vector Field**

and f is called a **Potential** of **F**

Many Applications of the Line Integral

Work

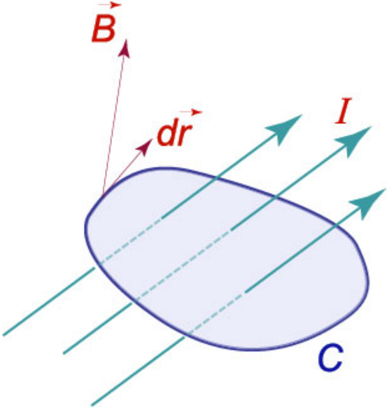
Position x along a line segment of a moving object is given by $x = g(t)$ where $g(0) =$ START and $g(T) =$ END.

$$\text{Work} = \int_{x=START}^{x=END} \mathbf{F}(x)\, dx = \int_0^T \mathbf{F}(g(t)) \cdot g'(t)\, dt$$

$x = g(t)$ implies $dx = g'(t)dt$

Other Physical Applications of Line Integrals

- Mass of a Wire
- Center of Mass and Moments of Inertia of a Wire;
- Magnetic Field Around a Conductor (**Ampere's Law)**: The line integral of a magnetic field **B** around a closed path C is equal to the total current flowing through the area bounded by the contour C

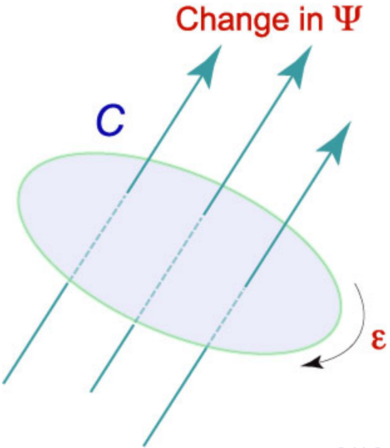


Other Physical Applications of Line Integrals

Voltage Generated in a Loop

(**Faraday's Law of Magnetic Induction**).

The electromotive force ϵ induced around a closed loop C is equal to the rate of the change of magnetic flux Ψ passing through the loop.



Applications in Economics

Buhr, Walter; Wagner, Josef

Working Paper

Line Integrals In Applied Welfare Economics: A Summary Of Basic Theorems

Volkswirtschaftliche Diskussionsbeiträge, No. 54-95

Provided in Cooperation with:

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