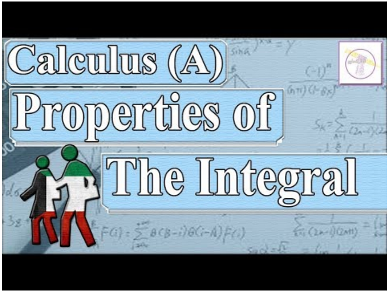
MATH 223: Multivariable Calculus



Class 24: April 15, 2022

Handouts

Multiple Integrals: Integration Theorems
Notes on Assignment 22
Assignment 23

Announcements

Review: Change of Variable (Method of Substitution)
Improper Integrals

Decide on Independent Project

This Week:

Definition of Multiple Integrals

Properties of the Integral

Change of Variable

Application to Probability

Example Evaluate $\int_{\mathcal{B}}(x^2+5y)dV$ where $0 \le x \le 1, 0 \le y \le 3$ **using the definition**.

Example Evaluate $\int_{\mathcal{B}}(x^2 + 5y)dV$ where $0 \leq x \leq 1, 0 \leq y \leq 3$ **using the definition**.

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x, y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Example Evaluate $\int_{\mathcal{B}}(x^2 + 5y)dV$ where $0 \leq x \leq 1, 0 \leq y \leq 3$ **using the definition**.

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x, y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be used.

Example Evaluate $\int_{\mathcal{B}}(x^2 + 5y)dV$ where $0 \leq x \leq 1, 0 \leq y \leq 3$ **using the definition**.

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x, y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be used.

For each $n = 1, 2, ...$ consider the Grid G_n consisting of the vertical lines $x = \frac{i}{n}, i = 0, 1, ..., n$ and the horizontal lines $y = \frac{j}{n}, j = 0, 1, ..., 3n$

Example Evaluate $\int_{\mathcal{B}}(x^2 + 5y)dV$ where $0 \leq x \leq 1, 0 \leq y \leq 3$ **using the definition**.

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x, y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be used.

For each $n = 1, 2, ...$ consider the Grid G_n consisting of the vertical lines $x = \frac{i}{n}, i = 0, 1, ..., n$ and the horizontal lines $y = \frac{j}{n}, j = 0, 1, ..., 3n$

Then mesh of $G_n = \frac{1}{n}$ and Area of Rectangle $R_{ij} = \frac{1}{n^2}$

Example Evaluate $\int_{\mathcal{B}}(x^2+5y)dV$ where $0 \le x \le 1, 0 \le y \le 3$ **using the definition**.

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x,y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be used.

For each $n = 1, 2, ...$ consider the Grid G_n consisting of the vertical lines $x = \frac{i}{n}, i = 0, 1, ..., n$ and the horizontal lines $y = \frac{j}{n}, j = 0, 1, ..., 3n$

Then mesh of $G_n = \frac{1}{n}$ and Area of Rectangle $R_{ij} = \frac{1}{n^2}$

Riemann sum is $\sum_{i=1}^{n} \left(\sum_{j=1}^{3n} \left[\left(\frac{i}{n}\right)^2 + 5\left(\frac{j}{n}\right) \right] \right) A(R_{ij})$

Riemann sum is $\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[\left(\frac{i}{n}\right)^2+5\left(\frac{j}{n}\right)\right]\right)A(R_{ij})$

$$=\frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{3n}\left(\frac{i}{n}\right)^2+\sum_{i=1}^{n}\sum_{j=1}^{3n}5\left(\frac{j}{n}\right)\right]$$

$$=\frac{1}{n^2}\left[3n\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2+n\sum_{j=1}^{3n}\frac{5j}{n}\right]$$

$$=\frac{1}{n^2}\left[\frac{3n}{n^2}\sum_{i=1}^{n}i^2+\frac{5n}{n}\sum_{j=1}^{3n}j\right]$$

$$=\frac{1}{n^2}\left[\frac{3}{n}\frac{n(n+1)(2n+1)}{6}+5\frac{(3n)(3n+1)}{2}\right]$$

Riemann sum is $\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[(\frac{i}{n})^2+5(\frac{j}{n})\right]\right)A(R_{ij})$

Riemann sum is $\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[(\frac{i}{n})^2+5(\frac{j}{n})\right]\right)A(R_{ij})$

$$= \frac{1}{n^2}\left[\frac{1}{2}(n+1)(2n+1)+\frac{15}{2}n(3n+1)\right]$$

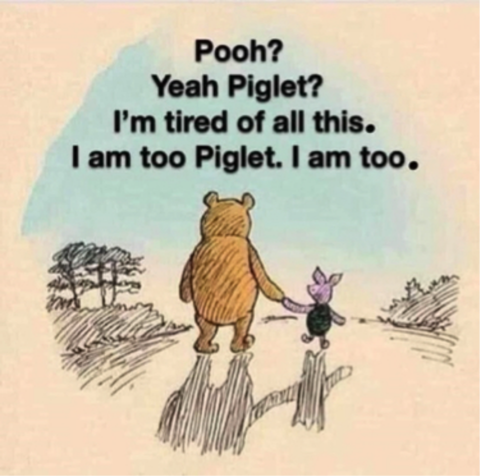
$$= \frac{1}{2}\left[(1+\frac{1}{n})(2+\frac{1}{n})\right]+\frac{15}{2}\left[3+\frac{1}{n}\right]$$

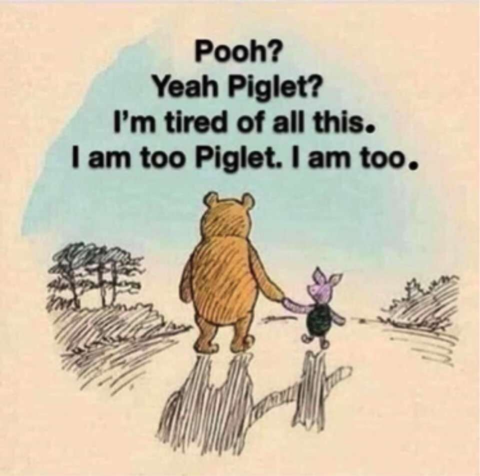
Riemann sum is $\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[(\frac{i}{n})^2 + 5(\frac{j}{n})\right]\right) A(R_{ij})$

$$= \frac{1}{n^2}\left[\frac{1}{2}(n+1)(2n+1) + \frac{15}{2}n(3n+1)\right]$$

$$= \frac{1}{2}\left[(1+\frac{1}{n})(2+\frac{1}{n})\right] + \frac{15}{2}\left[3+\frac{1}{n}\right]$$

Hence $\lim_{n\to\infty} = \frac{1}{2}(2) + \frac{15}{2}(3) = \frac{47}{2}$


Pooh?
Yeah Piglet?
I'm tired of all this.
I am too Piglet. I am too.



There Must Be a Better Way!

Evaluate As Iterated Integral

Evaluate As Iterated Integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=3} (x^2 + 5y)\, dy dx$$

Evaluate As Iterated Integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=3} (x^2 + 5y) dydx$$

$$= \int_{x=0}^{x=1} \left[x^2 y + \frac{5}{3} y^2 \right]_{y=0}^{y=3} dx$$

Evaluate As Iterated Integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=3} (x^2 + 5y) dy dx$$

$$= \int_{x=0}^{x=1} \left[x^2 y + \frac{5}{3} y^2 \right]_{y=0}^{y=3} dx$$

$$= \int_0^1 3x^2 + \frac{45}{2} dx = \left[x^3 + \frac{45}{2} x \right]_0^1 = (1 + \frac{45}{2}) - (0 + 0) = \frac{47}{2}$$

MULTIPLE INTEGRAL

Definition A function f is **integrable** over a bounded set $\mathcal{B}$ if there is a number $\int_{\mathcal{B}} fdV$ such that $\lim_{mesh(g)\to 0} \sum f(\vec{x_i})v(R_i)) = \int_{\mathcal{B}} fdV$ for every grid G covering $\mathcal{B}$ with mesh (G) and any choice of $\vec{x_i}$ in $\mathcal{R}_\rangle$

MULTIPLE INTEGRAL

Definition A function f is **integrable** over a bounded set $\mathcal{B}$ if there is a number $\int_{\mathcal{B}} fdV$ such that $\lim_{mesh(g)\to 0} \sum f(\vec{x_i})v(R_i)) = \int_{\mathcal{B}} fdV$ for every grid G covering $\mathcal{B}$ with mesh (G) and any choice of $\vec{x_i}$ in $\mathcal{R}_\rangle$

What This Limit Statement Means: For every $\epsilon > 0$, there is a $\delta > 0$ such that if G is a grid of mesh $< \delta$, then $|\int_{\mathcal{B}} fdV - \sum f(\vec{x_i})v(R_i))| < \epsilon$.

MULTIPLE INTEGRAL

Definition A function f is **integrable** over a bounded set $\mathcal{B}$ if there is a number $\int_{\mathcal{B}} fdV$ such that $\lim_{mesh(g)\to 0} \sum f(\vec{x_i})v(R_i)) = \int_{\mathcal{B}} fdV$ for every grid G covering $\mathcal{B}$ with mesh (G) and any choice of $\vec{x_i}$ in $\mathcal{R}_\rangle$

What This Limit Statement Means: For every $\epsilon > 0$, there is a $\delta > 0$ such that if G is a grid of mesh $< \delta$, then $|\int_{\mathcal{B}} fdV - \sum f(\vec{x_i})v(R_i))| < \epsilon$.

Theorem (not proved): $\int_{\mathcal{B}} fdV$ can be evaluated by Iterated Integrals.

Properties of the Integral

Linearity

Suppose f and g are both integrable over $\mathcal{B}$ while a and b are any real numbers.

Properties of the Integral

Linearity

Suppose f and g are both integrable over $\mathcal{B}$ while a and b are any real numbers.

Then $af + bg$ is integrable over $\mathcal{B}$ and

$$\int_{\mathcal{B}}(af + bg)dV = a\int_{\mathcal{B}} fdV + b\int_{\mathcal{B}} gdV$$

Properties of the Integral

Linearity

Suppose f and g are both integrable over $\mathcal{B}$ while a and b are any real numbers.

Then $af + bg$ is integrable over $\mathcal{B}$ and

$$\int_{\mathcal{B}} (af + bg)dV = a \int_{\mathcal{B}} f dV + b \int_{\mathcal{B}} g dV$$

Corollary (1) The set $\mathcal{V}$ of functions integrable over $\mathcal{B}$ is closed under addition and scalar multiplication so $\mathcal{V}$ is a vector space.

Properties of the Integral

Linearity

Suppose f and g are both integrable over $\mathcal{B}$ while a and b are any real numbers.

Then $af + bg$ is integrable over $\mathcal{B}$ and

$$\int_{\mathcal{B}}(af + bg)dV = a\int_{\mathcal{B}} fdV + b\int_{\mathcal{B}} gdV$$

Corollary (1) The set $\mathcal{V}$ of functions integrable over $\mathcal{B}$ is closed under addition and scalar multiplication so $\mathcal{V}$ is a vector space.
(2) The function $L : \mathcal{V} \to \mathbb{R}^1$ given by $L(f) = \int_{\mathcal{B}} fdV$ is a linear transformation.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that if S_1 and S_2 are Riemann sums for f and g respectively with mesh $< \delta$, then $|a||S_1 - \int_{\mathcal{B}} f dV| < \frac{\epsilon}{2}$ and $|b||S_2 - \int_{\mathcal{B}} g dV| < \frac{\epsilon}{2}$.

Now let S be a Riemann sum for $af + bg$ with mesh of grid $< \delta$.

$$\begin{aligned}\text{Then } S &= \sum (af + bg) f(\vec{x_i}) V(R_i) \\ &= a \sum f(\vec{x_i}) V(R_i) + b \sum g(\vec{x_i}) V(R_i) \\ &= aS_1 + bS_2\end{aligned}$$

Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that if S_1 and S_2 are Riemann sums for f and g respectively with mesh $< \delta$, then $|a||S_1 - \int_{\mathcal{B}} f dV| < \frac{\epsilon}{2}$ and $|b||S_2 - \int_{\mathcal{B}} g dV| < \frac{\epsilon}{2}$.

Now let S be a Riemann sum for $af + bg$ with mesh of grid $< \delta$.

$$\begin{aligned} \text{Then } S &= \sum (af + bg) f(\vec{x_i}) V(R_i) \\ &= a \sum f(\vec{x_i}) V(R_i) + b \sum g(\vec{x_i}) V(R_i) \\ &= aS_1 + bS_2 \end{aligned}$$

Now $|S - a\int f dV - b\int g dV| = |aS_1 - a\int f dV + bS_2 - b\int g dV|$
$\leq |a||S_1 - \int f dV| + |b||S_2 - \int g dV| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Theorem:(**Positivity**) If f is nonnegative and integrable over $\mathcal{B}$, then $\int_{\mathcal{B}} f dV \geq 0$.

Theorem:(**Positivity**) If f is nonnegative and integrable over $\mathcal{B}$, then $\int_{\mathcal{B}} f dV \geq 0$.

Theorem: If f, g are integrable on $\mathcal{B}$ with $f \geq g$, then $\int f \geq \int g$.

Theorem:(**Positivity**) If f is nonnegative and integrable over $\mathcal{B}$, then $\int_{\mathcal{B}} f dV \geq 0$.

Theorem: If f, g are integrable on $\mathcal{B}$ with $f \geq g$, then $\int f \geq \int g$.

Proof: $(f - g) \geq 0$ implies $\int_{\mathcal{B}} (f - g) dV \geq 0$

so $0 \leq \int_{\mathcal{B}} (f - g) dV = \int_{\mathcal{B}} f dV - \int_{\mathcal{B}} g dV$

Hence $\int_{\mathcal{B}} f dV \geq \int_{\mathcal{B}} g dV$

Theorem: If f and $|f|$ are integrable over $\mathcal{B}$, then

$$|\int_{\mathcal{B}} f dV| \leq \int_{\mathcal{B}} |f| dV$$

Theorem: If f and $|f|$ are integrable over $\mathcal{B}$, then
$$|\int_{\mathcal{B}} f dV| \leq \int_{\mathcal{B}} |f| dV$$

Proof: Start with $-|f| \leq f \leq |f|$

Then $-\int_{\mathcal{B}} |f| \leq \int_{\mathcal{B}} f \leq \int_{\mathcal{B}} |f|$

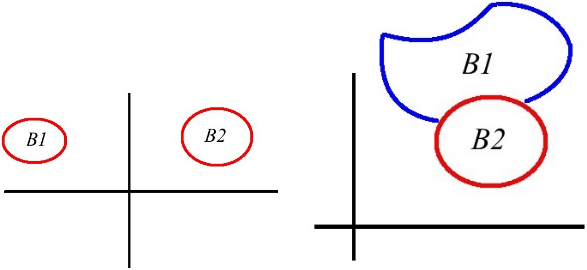
So $|\int_{\mathcal{B}} f| \leq \int_{\mathcal{B}} |f|$

Theorem (**Additivity**): If f is integrable over disjoint sets $\mathcal{B}_1$ and $\mathcal{B}_2$, then f is integrable over $\mathcal{B}_1 \cup \mathcal{B}_2$ with

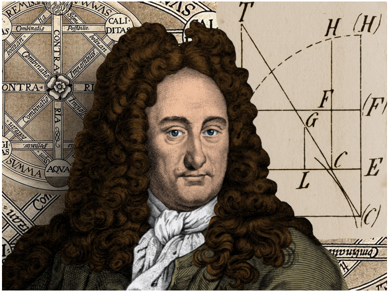
$$\int_{\mathcal{B}_1 \cup \mathcal{B}_2} f = \int_{\mathcal{B}_1} f + \int_{\mathcal{B}_2} f$$

<u>Theorem</u> (**Additivity**): If f is integrable over disjoint sets $\mathcal{B}_1$ and $\mathcal{B}_2$, then f is integrable over $\mathcal{B}_1 \cup \mathcal{B}_2$ with

$$\int_{\mathcal{B}_1 \cup \mathcal{B}_2} f = \int_{\mathcal{B}_1} f + \int_{\mathcal{B}_2} f$$



Leibniz Rule



Gottfried Wilhelm von Leibniz

July 1, 1646 – November 14, 1716

Biography

Leibniz Rule: Interchanging Differentiation and Integration

If g_y is continuous on $a \leq x \leq b, c \leq y \leq d$, then

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Example Compute $f(x) = \int_0^1 \frac{u^x - 1}{\ln u} du$

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Example Compute $f(x) = \int_0^1 \frac{u^x - 1}{\ln u} du$

By Leibniz:

$$f'(x) = \int_0^1 \frac{1}{\ln u}(u^x \ln u)du = \int_0^1 u^x du = \frac{u^{x+1}}{x+1}\Bigg|_{u=0}^{u=1} = \frac{1}{x+1}$$

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Example Compute $f(x) = \int_0^1 \frac{u^x - 1}{\ln u} du$
By Leibniz:

$$f'(x) = \int_0^1 \frac{1}{\ln u}(u^x \ln u)du = \int_0^1 u^x du = \frac{u^{x+1}}{x+1}\bigg|_{u=0}^{u=1} = \frac{1}{x+1}$$

So $f(x) = \ln(x+1) + C$ for some constant C.

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Example Compute $f(x) = \int_0^1 \frac{u^x - 1}{\ln u} du$

By Leibniz:

$$f'(x) = \int_0^1 \frac{1}{\ln u}(u^x \ln u)du = \int_0^1 u^x du = \frac{u^{x+1}}{x+1}\bigg|_{u=0}^{u=1} = \frac{1}{x+1}$$

So $f(x) = \ln(x+1) + C$ for some constant C.

To Find C, evaluate at $x = 0$:

$f(0) = \int_0^1 \frac{u^0 - 1}{\ln u} du = \int_0^1 0 = 0$

But $f(0) = \ln(0+1) + C = \ln(1) + C = 0 + C = C$ so $C = 0$ and

$f(x) = \ln(x+1)$

Example: Find $f'(y)$ if $f(y) = \int_0^1 (y^2 + t^2) dt$

Example: Find $f'(y)$ if $f(y) = \int_0^1 (y^2 + t^2) dt$

Method I: $f(y) = \int_0^1 (y^2 + t^2) dt = \left(y^2 t + \frac{t^3}{3}\right)\Big|_{t=0}^{t=1} = y^2 + \frac{1}{3}$ so

$$f'(y) = 2y$$

Example: Find $f'(y)$ if $f(y) = \int_0^1 (y^2 + t^2)dt$

Method I: $f(y) = \int_0^1 (y^2 + t^2)dt = (y^2 t + \frac{t^3}{3})\Big|_{t=0}^{t=1} = y^2 + \frac{1}{3}$ so

$$f'(y) = 2y$$

Method II: (Leibniz) $f'(y) = \int_0^1 2y dt = 2yt\Big|_0^1 = 2y$

Proof of Leibniz Rule

To Show:

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Proof of Leibniz Rule

To Show:

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Let $f(y) = \int_a^b g(x,y)dx$ and Use Definition of Derivative

$$f'(y) = \lim_{h\to 0} \frac{f(y+h)-f(y)}{h}$$

$$\frac{f(y+h)-f(y)}{h} = \frac{\int_a^b g(x,y+h)dx - \int_a^b g(x,y)dx}{h} = \frac{\int_a^b (g(x,y+h)-g(x,y))dx}{h}$$

$$f'(y) = \lim_{h \to 0} \frac{f(y+h) - f(y)}{h} = \lim_{h \to 0} \frac{\int_a^b [g(x, y+h) - g(x, y)]\, dx}{h}$$

Interchange Limit and Integral:

$$= \int_a^b \left(lim_{h \to 0} \frac{[g(x, y+h) - g(x, y)]}{h} \right) dx$$

$$= \int_a^b \frac{\partial g}{\partial y}(x, y) dx$$

Alternate Proof of Leibniz Rule

(**Uses Iterated Integral**)

Begin with $\int_a^b g_y(x,y)dx$

Let $I = \int_c^y (\int_a^b g_y(x,y)dx)dy$

Switch Order of Integration: $I = \int_a^b \left(\int_c^y g_y(x,y)dy\right) dx$

$$I = \int_a^b g(x,y)\Big|_{y=c}^{y=y} dx = \int_a^b g(x,y) - g(x,c)dx$$

$$= \int_a^b g(x,y)dx - \int_a^b g(x,c)dx$$

The left term is a function of y and the second is a constant C

Alternate Proof of Leibniz Rule (Continued)

$$I = \int_c^y \left(\int_a^b g_y(x,y)dx \right) dy = \int_a^b g(x,y)dx - C$$

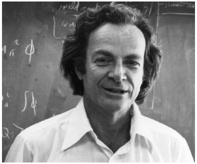
Now Take the Derivative of Each Side with Respect to y, using the Fundamental Theorem of Calculus on the left:

$$\int_a^b g_y(x,y)dx = \frac{d}{dy} \int_a^b g(x,y)dx - 0$$

Richard Feynman

May 11, 1918 – February 15, 1988

Nobel Prize in Physics, 1965



"I used that one damn tool again and again."

" I caught on how to use that method, and I used that one damn tool again and again. [If] guys at MIT or Princeton had trouble doing a certain integral, [then] I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me. (*Surely You're Joking, Mr. Feynman!*)

Richard Feynman's Integral Trick