MATH 223: Multivariable Calculus

Class 23: April 13, 2022

Multiple Integrals

Notes on Assignment 21

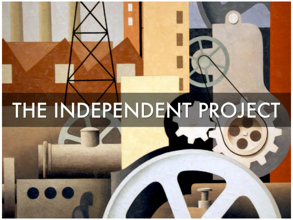
Assignment 22

Notes on Exam 2 (Median: 84)

Activity	Date	Approximate Weight
Exam 3	May 2	20%
Project	May 16	15%
Final Exam	May 19	25%

Announcements

Independent Projects



Choose Topic: Monday, April 18
Due: Monday, May 16

The Week Ahead:

Iterated Integral (Last Time)
Definition of Multiple Integrals
Properties of the Integral
Change of Variable

Instances of the Integral: I

The Classic Case

$$f := \mathbb{R}^1 \to \mathbb{R}^1$$

Definite Integral $\displaystyle\int_a^b f(x)dx$

Indefinite Integral $\displaystyle\int f(x)dx$

Instances of the Integral: II

Vector-Valued Functions of a Real Variable

$$f := \mathbb{R}^1 \to \mathbb{R}^n$$

$$f(t) = [f_1(t), f_2(t), ..., f_n(t)]$$

$$\int f(t) = \left[\int f_1(t), \int f_2(t), ..., \int f_n(t)\right]$$

Instances of the Integral: III

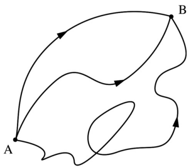
Line Integral = Path Integral

(Will Study in Chapter 7)

γ is graph of $g : \mathbb{R}^1 \to \mathbb{R}^n, a \leq t \leq b$

Force Field $F : \mathbb{R}^n \to \mathbb{R}^n$

$$\int_\gamma F = \int_a^b F(g(t)) \cdot g'(t) dt$$

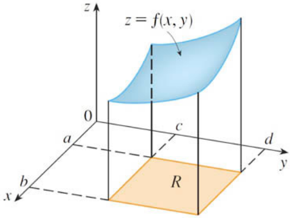


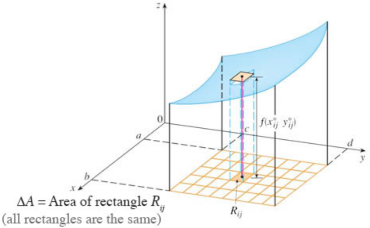
Instances of the Integral: IV

Iterated Integral

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$

$$\int_a^b \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy \right) dx$$





Instances of the Integral: V

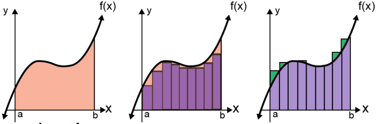
The Multiple Integral

A Generalization of Situation 1

$$\int_a^b f(x)dx = \lim_{max(\Delta(x_i)\rightarrow 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

$$\Delta x_i = \text{ length of ith subdivision}$$

$$x_i = \text{ any point in ith subinterval}$$



Area of region

Lower Sum

Upper Sum

Instances of the Integral: V

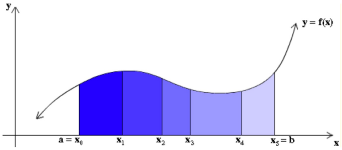
Unequal Divisions

$$\int_a^b f(x)dx = \lim_{max(\Delta(x_i)\to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

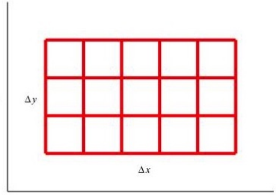
$$\Delta x_i = \text{ length of ith subdivision}$$

$$x_i = \text{ any point in ith subinterval}$$

Riemann Sums of Unequal Length Subintervals



First Extension: $f : \mathbb{R}^2 \to \mathbb{R}^1$ on a rectangle $\mathcal{R}$



Cover $\mathcal{R}$ with a grid G of horizontal and vertical lines
$mesh(G) = m(G)$=maximum length of edge of interval in grid.
Number Rectangles $R_1, R_2, ..., R_k$ with area of Rectangle R_i denoted by $A(R_i)$.
Pick a point $\vec{x_i}$ in R_i.

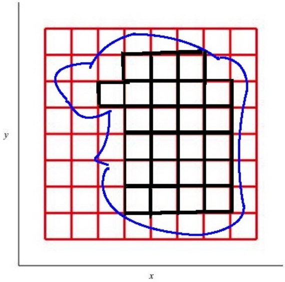
Form $\sum_{i=1}^{k} f(\vec{x_i})A(R_i)$; Take $\lim_{m(G)\to 0} \sum_{i=1}^{k} f(\vec{x_i})A(R_i)$

The limit is the integral of f over $\mathcal{R}$ and is denoted $\int_{\mathcal{R}} fdA$

Second Extension: $f : \mathbb{R}^2 \to \mathbb{R}^1$ on a BOUNDED set $\mathcal{B}$

Cover $\mathcal{B}$ with a grid G of horizontal and vertical lines

Let $R_1, R_2, ..., R_k$ be all bounded rectangles formed by G that lie **inside** $\mathcal{B}$.



Choose $\vec{x_i}$ in R_i.

$$\text{Take} \quad \lim_{m(G)\to 0} \sum_{i=1}^{k} f(\vec{x_i}) A(R_i)$$

The limit is the integral of f over $\mathcal{B}$ and is denoted $\int_{\mathcal{B}} f dV$

Theorem: If $\int_{\mathcal{B}} f dV$ exists and iterated integrals exist for some orders of partial integration, then all of these integrals are equal.

Proof:
Apostol, *Mathematical Analysis*
Sprivak, *Calculus on Manifolds*

When Does The Integral Exist?

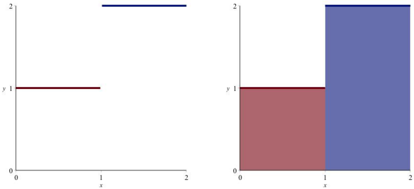
Idea: f does not have too many points of discontinuity

Definition: A set S has **zero content** if $\int_S 1dV = 0$.

Theorem: Let $\mathcal{B}$ be a bounded set in $\mathbb{R}^n$ whose boundary has zero content. Let f be a bounded function bounded on $\mathcal{B}$. If f is continuous on $\mathcal{B}$ except perhaps on a set of zero content, then $\int_{\mathcal{B}} fdV$ exists.

Example from Calculus 1

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 2 & 1 < x \leq 2 \end{cases}$$



$$\int_1^2 f(x)dx = 3$$

Generalize For a Function $f : \mathbb{R}^n \to \mathbb{R}^1$
Coordinate Rectangle in $\mathbb{R}^n$

$\mathcal{R} = \{(x_1, x_2, ..., x_n) : a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, ..., a_n \leq x_n \leq b_n\}$

Volume or Content of $\mathcal{R}$

$V(\mathcal{R}) = (b_1 - a_a)(b_2 - a_2)...(b_n - a_n)$

Grid: a finite set of $n - 1$ =dimensional planes in $\mathbb{R}$ parallel to the coordinate planes.

G divides $\mathbb{R}$ into a finite number of bounded "rectangles' $\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_k$ and possibly other unbounded "rectangles,

The mesh $m(G)$ of a grid = maximum "length" of a side of the rectangles $\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_k$

A set $\mathcal{B}$ is bounded if it can be covered by a grid.

$$\text{Then } \int_{\mathcal{B}} f dV = \lim_{m(G) \to 0} \sum_{i=1}^{k} f(\vec{x_i}) V(R_i)$$

if this limit exists. for all grids and all choices of $\vec{x_i}$ in R_i.

$$\text{Then } \int_{\mathcal{B}} f dV = \lim_{m(G)\to 0} \sum_{i=1}^{k} f(\vec{x_i})V(R_i)$$

if this limit exists. for all grids and all choices of $\vec{x_i}$ in R_i.

$$\text{Content of } \mathcal{B} = \int_{\mathcal{B}} 1 dV = \begin{cases} \text{Length of } \mathcal{B} & \text{if } \mathcal{B} \subset R^1, \\ \text{Area of } \mathcal{B} & \text{if } \mathcal{B} \subset R^2 \\ \text{Volume of } \mathcal{B} & \text{if } \mathcal{B} \subset R^3 \end{cases}$$

Example Evaluate $\int_{\mathcal{B}}(x^2 + 5y)dV$ where $0 \leq x \leq 1, 0 \leq y \leq 3$ **using the definition**.

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x, y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be used.

For each $n = 1, 2, ...$ consider the Grid G_n consisting of
the vertical lines $x = \frac{i}{n}, i = 0, 1, ..., n$ and
the horizontal lines $y = \frac{j}{n}, j = 0, 1, ..., 3n$

Then mesh of $G_n = \frac{1}{n}$ and Area of Rectangle $R_{ij} = \frac{1}{n^2}$

Riemann sum is $\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[\left(\frac{i}{n}\right)^2+5\left(\frac{j}{n}\right)\right]\right)A(R_{ij})$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{3n}\left(\frac{i}{n}\right)^2+\sum_{i=1}^{n}\sum_{j=1}^{3n}5\left(\frac{j}{n}\right)\right]$$

$$= \frac{1}{n^2}\left[3n\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2+n\sum_{j=1}^{3n}\frac{5j}{n}\right]$$

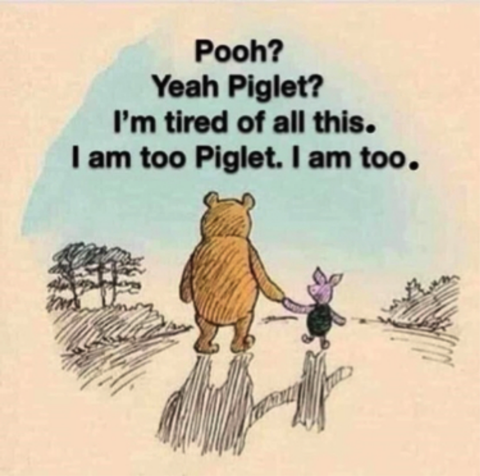
$$= \frac{1}{n^2}\left[\frac{3n}{n^2}\sum_{i=1}^{n}i^2+\frac{5n}{n}\sum_{j=1}^{3n}j\right]$$

$$= \frac{1}{n^2}\left[\frac{3}{n}\frac{n(n+1)(2n+1)}{6}+5\frac{(3n)(3n+1)}{2}\right]$$

Riemann sum is $\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[(\frac{i}{n})^2+5(\frac{j}{n})\right]\right)A(R_{ij})$

$$= \frac{1}{n^2}\left[\frac{1}{2}(n+1)(2n+1)+\frac{15}{2}n(3n+1)\right]$$
$$= \frac{1}{2}\left[(1+\frac{1}{n})(2+\frac{1}{n})\right]+\frac{15}{2}\left[3+\frac{1}{n}\right]$$

Hence $\lim_{n\to\infty} = \frac{1}{2}(2)+\frac{15}{2}(3)=\frac{47}{2}$



There Must Be a Better Way!

Evaluate As Iterated Integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=3} (x^2 + 5y) dy dx$$

$$= \int_{x=0}^{x=1} \left[x^2 y + \frac{5}{3} y^2 \right]_{y=0}^{y=3} dx$$

$$= \int_0^1 3x^2 + \frac{45}{2} dx = \left[x^3 + \frac{45}{2} x \right]_0^1 = (1 + \frac{45}{2}) - (0 + 0) = \frac{47}{2}$$