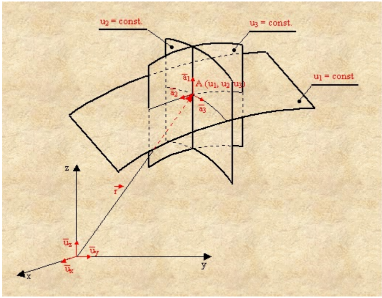
MATH 223: Multivariable Calculus



Class 21: April 8, 2022

Notes on Assignment 20

Assignment 21

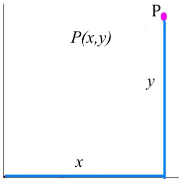
Curvilinear Coordinates

Announcements

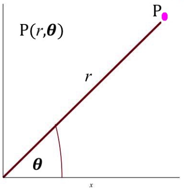
Exam 2: Monday
7 PM –?
Axinn 229

Review Basic Theorems About Integration from Calculus I

Today:
Curvilinear Coordinates
Coordinate Systems in Plane and Space
Plane



Cartesian



Polar

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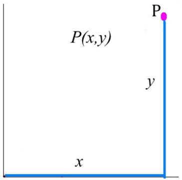
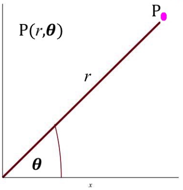


Cartesian



Polar

Newton(1671) [Not published until after death]
Jacob Bernoulli (1691)

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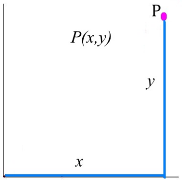

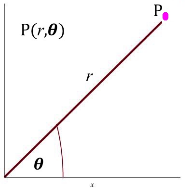

Cartesian **Polar**

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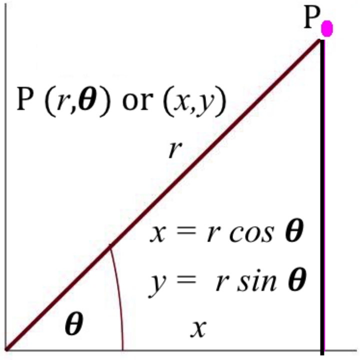
Jacob Bernoulli (1691)

In Polar Coordinates, Circles and Lines Through Origin Have Simple Equations:

Circle: $r = 4$

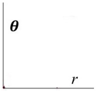
Line: $\theta = \pi/6$

Relationship Between Polar and Cartesian

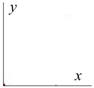


Linear Algebra Perspective

$$P\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ for } \begin{matrix} 0 < r < \infty \\ 0 \le \theta \le 2\pi \end{matrix}$$



$$P : U^2 \to \mathbb{R}^2$$



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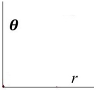

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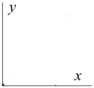
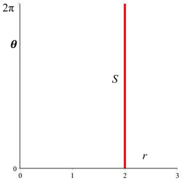
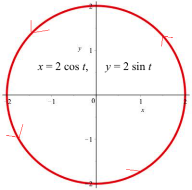


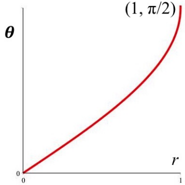
Example: P sends this line segment S to circle of radius 2 with center at origin.





$[r, \theta] = [2, t], 0 \le t \le 2\pi$

Example: $\theta = t, r = \sin t = \sin\theta$



	$r = \sin t$	$\theta = t$
0	0	0
$\pi/6$	$1/2$	$\pi\ /6$
$\pi/4$	$\sqrt{2}/2$	$\pi/4$
$\pi/2$	1	$\pi/2$

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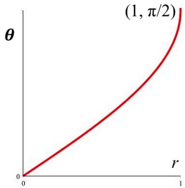

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Then $x = r\cos\theta = \sin\theta\cos\theta$ and $y = r\sin\theta = \sin\theta\sin\theta$

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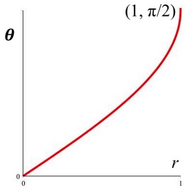

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Then $x = r\cos\theta = \sin\theta\cos\theta$ and $y = r\sin\theta = \sin\theta\sin\theta$
So $x^2 = \sin^2\theta\cos^2\theta, y^2 = \sin^2\theta\sin^2\theta$
and then $x^2 + y^2 = \sin^2\theta(\cos^2\theta + \sin^2\theta) = \sin^2\theta \times 1 = \sin^2\theta = y$

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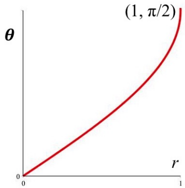

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Thus $x^2 + y^2 - y = 0$.

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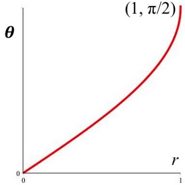

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Thus $x^2 + y^2 - y = 0$. Complete the square in y:

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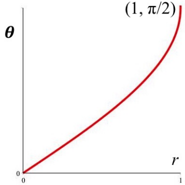

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Thus $x^2 + y^2 - y = 0$. Complete the square in y:

$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \implies x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

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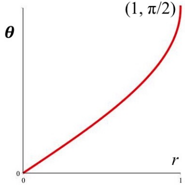

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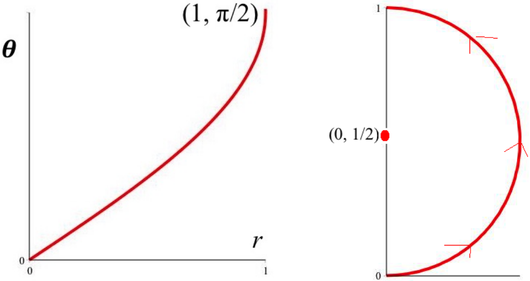
The image is the right half of the circle:



Think of P as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then

$$P' = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{pmatrix}$$

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$$P' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \implies P'(\pi/6) = \begin{pmatrix} \sqrt{3}/2 & -1/4 \\ 1/2 & \sqrt{3}/4 \end{pmatrix}$$

$$\frac{\text{Previous Example}}{g(t) = [\sin t, t]}$$

$$\text{so } g'(t) = [\cos t, 1]$$

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$g : \mathbb{R}^1 \to \mathbb{U}^2$ and $P : \mathbb{U}^2 \to \mathbb{R}^2$

$$(P \circ g) : \mathbb{R}^1 \to \mathbb{R}^2$$

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$$(P \circ g)' = P'(g) \cdot g'$$

Evaluate at $\pi/6$:

Think of P as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then

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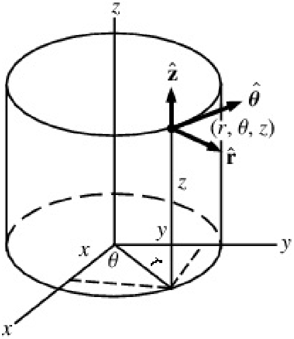
Evaluate at $\pi/6$: $\begin{pmatrix} \sqrt{3}/2 & -1/4 \\ 1/2 & \sqrt{3}/4 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$

Coordinate Systems in 3-Space

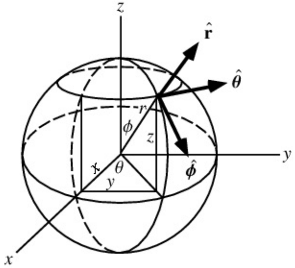
Cylindrical Coordinates: (r, θ, z).



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Coordinate Systems in 3-Space

Spherical Coordinates: $(\rho, \theta, \phi) = (r, \theta, \phi)$



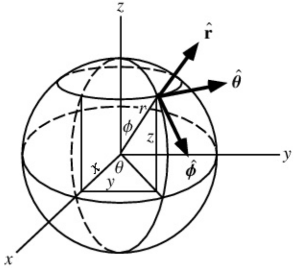
$r =$ distance between origin and point

$\theta =$ project down to xy-plane

$\phi =$ rotation down from vertical axis

Coordinate Systems in 3-Space

Spherical Coordinates: $(\rho, \theta, \phi) = (r, \theta, \phi)$



$r =$ distance between origin and point

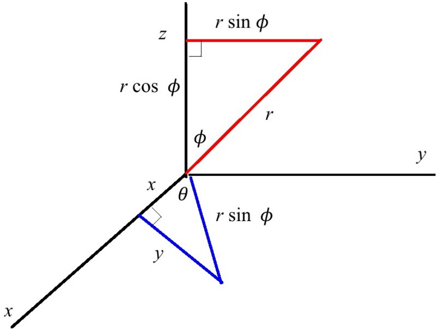
$\theta =$ project down to xy-plane

$\phi =$ rotation down from vertical axis

$r =$ distance between origin and point	$x = r \sin\phi \cos\theta$
$\theta =$ project down to xy-plane.	$y = r \sin\phi \sin\theta$
$\phi =$ rotation down from vertical axis	$z = r \cos\phi$

Some authors use ρ instead of r.

Converting from Spherical To Cartesian

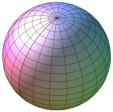


$$\cos \phi = \frac{z}{r} \implies z = r \cos \phi$$

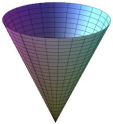
$$\cos \theta = \frac{x}{r \sin \phi} \implies x = r \sin \phi \cos \theta$$

$$\sin \theta = \frac{y}{r \sin \phi} \implies y = r \sin \phi \sin \theta$$

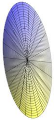
$$S : \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \to \begin{pmatrix} r \sin\phi \cos\theta \\ r \sin\phi \sin\theta \\ r \cos\phi \end{pmatrix}$$



$r =$ Constant
Sphere



$\phi =$ Constant
Cone



$\theta =$ Constant
Plane

Jacobian Matrices

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}$$