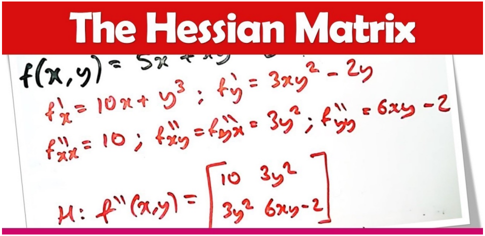
# MATH 223: Multivariable Calculus




Class 20: April 6, 2022

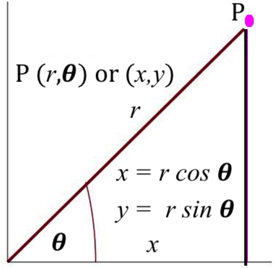
Notes on Assignment 18

Assignment 20

Ludwig Otto Hesse

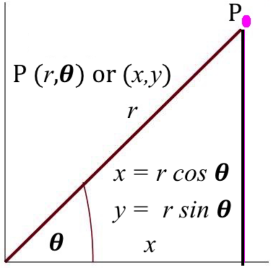
ANNOUNCEMENTS

# Review Polar Coordinates



ANNOUNCEMENTS

## Review Polar Coordinates




**Exam 2: Monday, April 11**

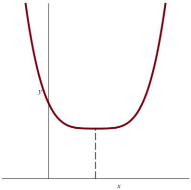
**Friday's Class on Zoom**

Today:

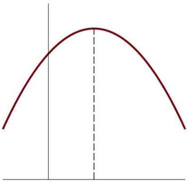
**Second Derivative Criteria**

Classic Case: $f : \mathbb{R}^1 \to \mathbb{R}^1$

with $f'(x_0) = 0$




$f''(x_0) > 0$



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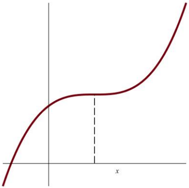


$f''$ changes sign at $x_0$

# What Can We Conclude if $f'(x_0) = 0$?

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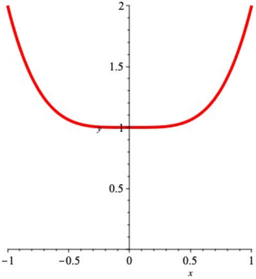
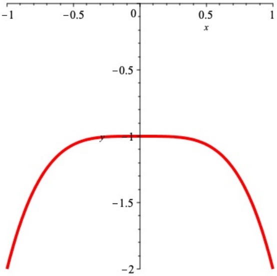



$$f(x) = x^4$$
$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$
$$f''(0) = 0$$




$$f(x) = -x^4$$
$$f'(x) = -4x^3$$
$$f''(x) = -12x^2$$
$$f''(0) = 0$$

## Second. Derivative Test
## For Real-Valued Functions of Several Variables

Second. Derivative Test
For Real-Valued Functions of Several Variables

**Involves Second Order Partial Derivatives**

Second. Derivative Test

For Real-Valued Functions of Several Variables

# Involves Second Order Partial Derivatives

**Definition**: If $f$ is a twice differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^1$, then the **Hessian Matrix** is the $n \times n$ matrix of second order partial derivatives of $f$

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For example, if $f : \mathbb{R}^3 \to \mathbb{R}^1$ so $w = f(x, y, z)$, then the Hessian $\mathcal{H}$ for $f$ is

$$\mathcal{H}(f) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

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Note that if $f$ is twice continuously differentiable, then the mixed partials are equal: $f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$ so the Hessian matrix is symmetric.

Otto Ludwig Hesse
April 22, 1811 – August 4, 1874

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for real-valued functions of several variables
replaces the condition $f''(c)$ being positive
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for real-valued functions of several variables
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It similarly uses the **negative definite** character of the Hessian matrix in place of the negativity of the second derivative.

**Definition** A *Positive Definite Matrix* is an $n$ by $n$ symmetric matrix $A$ such that $\mathbf{x} \cdot (A\mathbf{x}) > 0$ for all nonzero vectors $\mathbf{x}$ in $\mathbb{R}^n$.

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If the strict inequality sign $>$ is replaced by the weaker $\geq$, then the matrix is called *positive semi-definite*.

We define *negative definite* and *negative semi-definite* in an analogous manner, using $<$ and $\leq$, respectively.

**Example**: Let $A = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$.

With $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, we have

$$\begin{aligned}
(x, y) \cdot (10x + 4y, 4x + 2y) &= 10x^2 + 4xy + 4xy + 2y^2 \\
&= 10x^2 + 8xy + 2y^2 \\
&= (9x^2 + 6xy + y^2) + (x^2 + 2xy + y^2) \\
&= (3x + y)^2 + (x + y)^2
\end{aligned}$$

which is the sum of non-negative numbers and hence always greater than or equal to 0, but is positive unless both $x$ and $y$ are 0. Hence $A$ is a positive definite matrix.

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*Proof:* Let $A$ be an $n \times n$ be a positive definite matrix with real eigenvalue $\lambda$.

If $\lambda = 0$, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. But then, $\mathbf{x} \cdot (A\mathbf{x}) = 0$ so $A$ would not be positive definite. □

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If $\lambda < 0$, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ in which case

$$\mathbf{x} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (\lambda\mathbf{x}) = \lambda|\mathbf{x}|^2 < 0$$

so again $A$ is not positive definite. □

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so again $A$ is not positive definite. □Not only is the converse of this theorem true (all eigenvalues positive implies positive definiteness), but the eigenvalues of a symmetric matrix are always real.

## Theorem

*If all the eigenvalues of a symmetric matrix $A$ are positive, then $A$ is positive definite.*

*Proof*: We make use of a result from linear algebra: A symmetric matrix is diagonalizable by an orthogonal matrix; that is, there is an orthogonal matrix $Q$ such that $Q^T = Q^{-1}$ with $Q^TAQ = D$, where $D$ is a diagonal matrix whose main diagonal entries are the eigenvalues of $A$:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Let $\mathbf{x}$ be any nonzero vector and set $\mathbf{y} = Q^T\mathbf{x}$ so that $\mathbf{y}^T = \mathbf{x}^T Q$. Then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(QDQ^T)\mathbf{x} = (\mathbf{x}^T(Q)D(Q^T\mathbf{x}^T) = \mathbf{y}^T D\mathbf{y}$$

but

$$\mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2 = \sum_{i=1}^{n} \lambda_i y_i^2$$

where $\mathbf{y}^T = (y_1, y_2, ..., y_n)$. Since $\mathbf{x}$ is a nonzero vector and $Q$ is invertible, at least one $y_i$ is nonzero. Hence

$$\mathbf{x}^T A\mathbf{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2 = \sum_{i=1}^{n} \lambda_i y_i^2$$

is the sum of non-negative terms, at least one of which is positive, so it is positive. Thus $A$ is positive definite. $\square$.

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The Second-Order Taylor Theorem asserts that

$$f(\mathbf{x_0} + \mathbf{h}) = f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^T \mathcal{H}(\mathbf{x_0})\mathbf{h} + R_2(\mathbf{x_0}, \mathbf{h})$$

and

$$\lim_{\mathbf{h} \to 0} \frac{R_2(\mathbf{x_0}, \mathbf{h})}{|\mathbf{h}|} = 0$$

(See Text for Proof)

Theorem

**Second Derivative Test for Local Extrema**.

*Suppose* $f : \mathbb{R}^n \to \mathbb{R}^1$ *has continuous third order partial derivatives on a neighborhood of* $\mathbf{x_0}$ *which is a critical point of* $f$ *.*

*IF the Hessian* $\mathcal{H}$ *evaluated at* $\mathbf{x_0}$ *is positive definite, then* $f$ *has a relative minimum at* $\mathbf{x_0}$*.*

*If the Hessian is negative definite, then there is a relative maximum at the critical point.*

Here is the idea of the proof: Since $\mathbf{x_0}$ is a critical point, $\nabla f(\mathbf{x_0}) = \mathbf{0}$ and by Taylor's Theorem

$$f(\mathbf{x_0} + \mathbf{h}) = f(\mathbf{x_0}) + \frac{1}{2}\mathbf{h}^T \mathcal{H}(\mathbf{x_0})\mathbf{h} + R_2(\mathbf{x_0}, \mathbf{h})$$

where the remainder term is negligible when $\mathbf{h}$ is very small. Thus

$$f(\mathbf{x_0} + \mathbf{h}) \approx f(\mathbf{x_0}) + \frac{1}{2}\mathbf{h}^T \mathcal{H}(\mathbf{x_0})\mathbf{h}.$$

If the Hessian is positive definite, then the second term is positive for $\mathbf{h} \neq \mathbf{0}$ so $f(\mathbf{x_0} + \mathbf{h}) > f(\mathbf{x_0})$ when $\mathbf{h}$ is sufficiently small, making $\mathbf{x_0}$ the location of a relative minimum.
We will leave a formal proof and dealing with the negative definite case for the exercises.

∎

Example: Our Temperature Function

$T(x, y) = 2x^2 + 4y^2 + 2x + 1$

Here $T_x(x, y) = 4x + 2$ and $T_y(x, y) = 8y$.

Thus, the Hessian Matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$$

whose eigenvalues are 4 and 8.

Both are positive so $T$ has a minimum wherever the gradient is 0; that is, at (-1/2,0).

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whose eigenvalues are 2 and -2.
Thus it is neither positive definite nor negative definite.

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$\mathbf{x} \cdot \mathcal{H}\mathbf{x}$ can be positive ($\mathbf{x} = (1, 0)$) or negative ($\mathbf{x} = (0,1)$) so there is a **saddle point** at any point where $\nabla T$ is $\vec{0}$.

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The first equation gives $9x^4 = 4y^2$ and the second yields $y^2 = -\frac{2}{3}x$
Thus $9x^4 = 4(-\frac{2}{3}x) = -\frac{8}{3}x$
so $9x^4 = -\frac{8}{3}x$ or $27x^4 + 8x = 0$; Hence $x(27x^3 + 8) = 0$
This gives two solutions: $x = 0, y = 0$ and $x = -\frac{2}{3}, y = \frac{2}{3}$

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whose eigenvalues are -2 and -6, both negative.

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Thus there is a relative maximum at (-2/3,2/3),/

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At (0,0), Hessian is

$$\mathcal{H} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

whose eigenvalues are -2 and $+2$.

. Thus there is a saddle point at (0,0),

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Look at the Eigenvectors!

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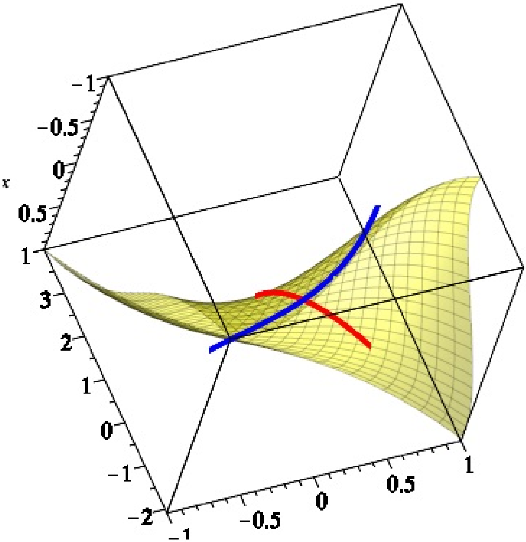
Consider $f(x, -x) = x^3 + x^3 + 2xx = 2x^3 + 2x^2 = 2x^2(1 + x)$ has relative minimum at $x = 0$

Graph of $f(x,y) = x^3 - y^3 - 2xy$



Next Time **Alternative Coordinate Systems for 3-Space**

*Rectangular*
*Cylindrical*
*Spherical*